Chapter 3: Fourier series

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Section 3.1 Piecewise Smooth Functions and Periodic Extensions
Section 3.2 Convergence of Fourier series
Section 3.3 Fourier cosine and sine series
Section 3.4 Term-by-term differentiation
Section 3.5 Term-by-term Integration
Section 3.6 Complex form of Fourier series
Outline

Section 3.1 Piecewise Smooth Functions and Periodic Extensions

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Piecewise smooth functions

Definition
A function $f : [a, b] \rightarrow \mathbb{R}$ is **piecewise continuous** if it is continuous on $[a, b]$ except at finitely many points. If both $f$ and $f'$ are piecewise continuous, then $f$ is called **piecewise smooth**.

- PC: may have finitely many jump discontinuity, but $f(x^-)$ and $f(x^+)$ exist for all $x \in [a, b]$.
- Are these functions PC or PS? Suppose that $x \in [-\pi, \pi]$:

<table>
<thead>
<tr>
<th>function</th>
<th>PC</th>
<th>PS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_1(x) = \sin(10x)$;</td>
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<tr>
<td>$f_2(x) =</td>
<td>x</td>
<td>$;</td>
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<tr>
<td>$f_3(x) = x^{1/3}$;</td>
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<td>$f_4(x) = 1_{[0,1]}(x)$</td>
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<tr>
<td>$f_5(x) = \begin{cases} -\ln(1-x), &amp; -\pi \leq x &lt; 1; \ 1, &amp; 1 \leq x \leq \pi \end{cases}$</td>
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**Periodic extension.** If $f$ is defined on $[-L, L]$, then its periodic extension is

$$
\tilde{f}(x) = \begin{cases} 
\vdots 
\quad f(x + 2L), & -3L < x < -L; \\
\quad f(x), & -L < xL; \\
\quad f(x - 2L), & L < x < 3L; \\
\quad \vdots 
\end{cases}
$$

- The end points?
- Example (how to make the extension in a sketch?)
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Convergence of Fourier series

\[ f(x) = a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \right) \]

▶ Convergent series?
- point-wise, almost everywhere, uniform.
- radius of convergence of \( g(x) = \sum_{n=1}^{\infty} a_n x^n \): \( r = \lim_{n \to \infty} \frac{a_n}{a_{n+1}} \).
- Weierstrass M-test: the series \( \sum_{n=1}^{\infty} f_n(x) \) converges uniformly in \( D \) if \( |f_n(x)| \leq c_n \) for \( x \in D \) and \( \sum_{n=1}^{\infty} c_n < \infty \).

▶ Is the limit \( f \)? A more precise notation:

\[ f(x) \sim a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \right) = \tilde{f}(x) \]

The Fourier coefficient of \( f \) (by orthogonality)

\[ a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) dx, \quad a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi}{L} x dx, \quad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi}{L} x dx \]
Theorem (Fourier Convergence Theorem)

If \( f \) is piecewise smooth on \([-L, L]\), then the Fourier series of \( f \) converges to

1. the periodic extension \( \tilde{f} \), at where \( \tilde{f} \) is continuous;
2. the average \( \frac{1}{2} [f(x^-) + f(x^+)] \) at where \( \tilde{f} \) has a jump discontinuity.

▶ Note: 2 includes 1. Together:

\[
\frac{1}{2} [f(x^-) + f(x^+)] = a_0 + \sum_{n=1}^{\infty} (a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x)
\]

▶ Proof: use Dirichlet kernel:
\[
D_N(x) = \frac{1}{2} + \sum_{n=1}^{N} \cos(nx) = \frac{\sin(N+\frac{1}{2})x}{2 \sin \frac{x}{2}}
\]

Notation: \( f \), periodic extension \( \tilde{f} \), Fourier series (limit) \( \widetilde{f}(x) \)
Sketch Fourier series Given $f$. Can we sketch the Fourier series $\tilde{f} = a_0 + \sum_{n=1}^{\infty} (a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x)$ without knowing $a_n, b_n$?

Yes! A simple application of the (powerful!) Fourier theorem: 3 steps

1. sketch $f$ on $[-L, L]$
2. Period extension of $f$ to $[-3L, 3L]$
3. sketch $\tilde{f}$: same as $\bar{f}$ except average at jumps

Example: $f(x) = \begin{cases} 0, & -L \leq x < L/2; \\ 1, & L/2 \leq x \leq L \end{cases}$

Q1: what if unbounded domain? $f(x) = \begin{cases} 0, & x < 0; \\ 1, & x \geq 0 \end{cases}$

Q2: half domain: $f(x)$ defined only for $x \in [0, L]$?

(Recall in HE+BC(Dirichlet/Neumann) + IC: $x \in [0, L]$)
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Fourier sine series

Fourier series of odd functions
When $f(x)$ on $[-L, L]$ is odd: $a_n = ? b_n = ?$

$$a_n = 0, b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x \, dx = B_n$$

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x$$

Fourier sine series: for $f(x)$ on $[0, L]$

$$f(x) \sim \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} x$$

Section 3.3 Fourier cosine and sine series
Sketch Fourier sine series Given \( f \), sketch the Fourier sine series
\[
\tilde{f} = \sum_{n=1}^{\infty} B_n \sin \frac{n \pi}{L} x
\]
without knowing \( B_n \)?

1. sketch \( f \) on \([0, L]\)
2. Odd periodic extension of \( f \) to \([-3L, 3L]\): \( f_{\text{odd}} \)
3. sketch \( \tilde{f} \): same as \( f_{\text{odd}} \) except average at jumps

Example: \( f(x) = 100, \ x \in [0, L] \)?
sketch:

Compute \( B_n \):
\[
B_n = \frac{2}{L} \int_{0}^{L} f(x) \sin \frac{n \pi}{L} x \, dx = \frac{200}{L} \int_{0}^{L} \sin \frac{n \pi}{L} x \, dx = \frac{400}{n \pi} 1_{\{n \text{ odd}\}}
\]

\[
100 = \sum_{n=1}^{\infty} B_n \sin \frac{n \pi}{L} x = \frac{400}{\pi} \left[ \sin \frac{\pi}{L} x + \frac{1}{3} \sin \frac{3 \pi}{L} x + \cdots \right], \quad x \in (0, L)
\]

► A series representation for \( \pi \):
\[
\frac{\pi}{4} = \sin \frac{\pi}{L} x + \frac{1}{3} \sin \frac{3 \pi}{L} x + \cdots \text{ for } x \in (0, L)
\]
at \( x = \frac{L}{2} \Rightarrow \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} + \cdots = \sum_{n=1}^{\infty} (-1)^{n} \frac{1}{n}
\]

► Equality holds on \( x \in (0, L) \), but not at \( x = 0, x = L \).

► Discontinuity: \( \tilde{f}(0) = 0, \tilde{f}(L) = 0 \), but \( f(x) = 100 \)
Physical example: HE+BC(Dirichlet) + IC: \( x \in [0, L] \)

\[
\begin{align*}
\partial_t u &= \kappa \partial_{xx} u, \quad \text{with } x \in (0, L), t > 0 \\
u(0, t) &= 0, u(L, t) = 0 \\
u(x, 0) &= f(x), \quad x \in [0, L]
\end{align*}
\]

Solution: IF

\[
f(x) \overset{\text{IF}}{=} \sum_{n=1}^{\infty} B_n \sin \left( \frac{n\pi}{L} x \right),
\]

\[
u(x, t) = \sum_{n=1}^{\infty} B_n \sin \left( \frac{n\pi}{L} x \right) e^{-\lambda_n \kappa t},
\]

- The equality does not hold! The series \( \tilde{f} \neq f \) at \( x = 0, x = L \).
- Physical meaning?
- numerical approximation \( \downarrow \)
Fourier series computation and the Gibbs Phenomenon

In numerical computation, we can only have finitely many terms.

\[ f(x) \approx f_N(x) = \sum_{n=1}^{N} B_n \sin \frac{n\pi x}{L} \]

For \( f(x) = 100, \ x \in [0, L] \), what will happen as \( N \to \infty \)?

- for \( x \in (0, L) \), \( f_N(x) \to f(x) \)
- \( f_N(0) \to \tilde{f}(0) = 0, \ f_N(L) \to \tilde{f}(L) = 0 \)
- Gibbs phenomenon: overshoot(undershoot) at the jump discontinuity

\[ \lim_{N \to \infty} f_N(0 + \frac{L}{2N}) \approx f(0^+) + [f(0^+) - f(0^-)] \ast 0.0895 \]
Fourier cosine series

Similar to sine series:

- When \( f(x) \) on \([-L, L]\) is EVEN: \( b_n = 0 \) → Fourier cosine series
- For \( f(x) \) on \([0, L]\), even extension → Fourier cosine series

\[
f(x) \sim \sum_{n=0}^{\infty} A_n \cos \frac{n\pi}{L} x
\]

- Odd periodic extension to sketch \( \tilde{f} \).
\( f(x) \) on \((0, L)\) by both sine and cosine series

Example: \( f(x) = \cos \frac{2\pi}{L} x \) on \( x \in (0, L) \)

Sine series: \( f(x) \sim \sum_{n=0}^{\infty} B_n \sin \frac{n\pi}{L} x \) with \( B_n = \frac{2}{L} \int_{0}^{L} f(x) \sin \frac{n\pi}{L} x \, dx \)

Cosine series: \( f(x) \sim \sum_{n=0}^{\infty} A_n \cos \frac{n\pi}{L} x \) with \( A_n = 0 \) if \( n \neq 2 \), \( A_2 = 1 \)

Even and odd parts

\[
    f(x) = f_{\text{even}}(x) + f_{\text{odd}}(x) = \frac{1}{2} [f(x) + f(-x)] + \frac{1}{2} [f(x) - f(-x)]
\]

\[
    \tilde{f}(x) = \tilde{f}_{\text{even}}(x) + \tilde{f}_{\text{odd}}(x) = \text{Cosine series + Sine Series}
\]
Continues Fourier Series

What condition on $f$ to make its Fourier series continuous?

Let $f$ be piecewise smooth, and denote its Fourier (sine/cosine) series by $\tilde{f}$.

- Fourier series $\tilde{f}$ is conti. and $\tilde{f} = f$ on $[-L, L]$ iff $f(-L) = f(L)$;

- Fourier sine series $\tilde{f}$ is conti. and $\tilde{f} = f$ on $[0, L]$ iff $f(0) = f(L) = 0$;

- Fourier cosine series $\tilde{f}$ is conti. and $\tilde{f} = f$ on $[-L, L]$ iff $f$ is conti.
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Question: can we exchange the order of the two operations:

$$\frac{d}{dx} \sum_{n=1}^{\infty} \sin n \pi \frac{x}{L} = \sum_{n=1}^{\infty} \frac{d}{dx} \sin n \pi \frac{x}{L}$$

Motivation: when solving PDE by separation of variables

$$\partial_t u = \kappa \partial_{xx} u, \text{ with } x \in (0, L), t > 0$$

$$u(0, t) = 0, u(L, t) = 0$$

$$u(x, 0) = f(x), \text{ } x \in [0, L]$$

We get

$$u(x, t) = \sum_{n=1}^{\infty} B_n \sin \left( \frac{n \pi x}{L} \right) e^{-\lambda_n \kappa t},$$

with $B_n$ determined by

$$f(x) = \sum_{n=1}^{\infty} B_n \sin \left( \frac{n \pi x}{L} \right) x.$$

To be addressed:

- Does the series converge?

$$\partial_t \sum_{n=1}^{\infty} \kappa \partial_{xx} \sum_{n=1}^{\infty}$$

$$\partial_t \sum_{n=1}^{\infty} \kappa \partial_{xx} \sum_{n=1}^{\infty}$$

$$\partial_t \sum_{n=1}^{\infty} = \sum_{n=1}^{\infty} \partial_t$$

$$\partial_{xx} \sum_{n=1}^{\infty} = \sum_{n=1}^{\infty} \partial_{xx}$$
**Example:** Consider Fourier series of \( f(x) = x, \ x \in [0, L] \):

- Find the Fourier series of \( f \)
- Try term by term Diff. (TBTD)

\[
x = \sum_{n=1}^{\infty} \frac{2L}{n\pi} (-1)^{n+1} \sin \frac{n\pi x}{L} =: \tilde{f}, \quad x \in (0, L)
\]

**TBTD:**

\[
1 = \sum_{n=1}^{\infty} 2(-1)^{n+1} \cos \frac{n\pi x}{L},
\]

at \( x = 0 \): the RHS = \( 2 \sum_{n=1}^{\infty} (-1)^{n+1} \) diverges!

\( \Rightarrow \) no TBTD

Q: \( f(x) = x \) is such a “good” function. What’s the problem?
Consider first Fourier sine series: $f$ odd; $f'$ even

$$f \text{ PC}, f' \text{ PC} \quad f(x) \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

$$f' \text{ PC}, f'' \text{ PC} \quad f'(x) \sim A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L}$$

If TBTD:

$$f'(x) \sim \sum_{n=1}^{\infty} b_n \frac{n\pi}{L} \cos \frac{n\pi x}{L},$$

which requires

$$A_0 = 0; A_n = b_n \frac{n\pi}{L}.$$ 

Thus (recall $b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx$)

$$0 = A_0 = \frac{1}{L} \int_0^L f'(x) \, dx = \frac{1}{L} [f(L) - f(0)] \quad \Rightarrow \quad f(L) = f(0)$$

$$A_n = \frac{2}{L} \int_0^L f'(x) \cos \frac{n\pi x}{L} \, dx =$$
TBTD of Fourier sine series \( f \) on \([0, L]\)

- \( f \) PS \(\Rightarrow\) its Fourier sine series converges:

\[
f(x) \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} = \frac{1}{2} [f(x^-) + f(x^+)]
\]

- \( f' \) PS, \(\Rightarrow\) Fourier series of \( f' \) converges
  if in addition, \( f \) continuous: \(\Rightarrow\)

\[
f'(x) \sim \frac{1}{L} [f(L) - f(0)] + \sum_{n=1}^{\infty} \left[ \frac{n\pi}{L} b_n + \frac{2}{L} (-1)^n [f(L) - f(0)] \right] \cos \frac{n\pi x}{L}
\]

- TBTD if \( f, f' \) are PS, \( f \) continuous and \( f(L) = f(0) = 0 \).
TBTD of Fourier cosine series $f$ on $[0, L]$

- $f$ PS $\Rightarrow$ its Fourier sine series converges:

$$f(x) \sim \sum_{n=0}^{\infty} a_n \cos \frac{n\pi x}{L} = \frac{1}{2} [f(x^-) + f(x^+)]$$

- $f' \text{ PS}, \Rightarrow$ Fourier series of $f'$ converges if in addition, $f$ continuous: $\Rightarrow$ (check it)

$$f'(x) \sim \sum_{n=1}^{\infty} \frac{n\pi}{L} a_n (-1) \sin \frac{n\pi x}{L}$$

- TBTD if $f, f'$ are PS, $f$ continuous.

Section 3.4 Term-by-term differentiation
TBTD of Fourier series $f$ on $[-L, L]$

- $f$ PS $\Rightarrow$ its Fourier series converges:

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} = \frac{1}{2} \left[f(x^-) + f(x^+)\right]$$

- $f'$ PS, $\Rightarrow$ Fourier series of $f'$ converges
  if in addition, $f$ continuous: $\Rightarrow$

$$f'(x) \sim$$

- TBTD if $f, f'$ are PS, $f$ continuous and $f(L) = f(-L)$. 

Section 3.4 Term-by-term differentiation
Back to PDE:
\[ \partial_t u = \kappa \partial_{xx} u, \text{ with } x \in (0, L), \, t > 0 \]
\[ u(0, t) = 0, \, u(L, t) = 0 \]
\[ u(x, 0) = f(x), \quad x \in [0, L] \]

We get
\[ u(x, t) = \sum_{n=1}^{\infty} B_n \sin \left( \frac{n\pi}{L} x \right) e^{-\lambda_n \kappa t}, \]

with \( B_n \) determined by
\[ f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} x. \]

To be addressed:

▶ Does the series converge?

\[ \partial_t \sum_{n=1}^{\infty} = \kappa \partial_{xx} \sum_{n=1}^{\infty} \]
\[ \partial_t \sum_{n=1}^{\infty} = \sum_{n=1}^{\infty} \partial_t \]
\[ \partial_{xx} \sum_{n=1}^{\infty} = \sum_{n=1}^{\infty} \partial_{xx} \]

▶ for each \( t \): \( u(x, t) \) is conti. & \( \partial_x u \) PS, BC \( \Rightarrow \) TBTD sine series

\( \partial_x u \) is conti. & \( \partial_{xx} u \) PS \( \Rightarrow \) TBTD cosine series

\[ \partial_{xx} \sum_{n=1}^{\infty} = \sum_{n=1}^{\infty} \partial_{xx} \]

▶ \( \partial_t u \) PS \( \Rightarrow \)
\[ \partial_t \sum_{n=1}^{\infty} = \sum_{n=1}^{\infty} \partial_t \]
Method of eigenfunction expansion (a generalization separation of variables) Seek solution of the form

\[ u(x, t) = \sum_{n=0}^{\infty} a_n(t) \cos \frac{n\pi}{L} x + b_n(t) \sin \frac{n\pi}{L} x, \]

- PDE + BC determines the eigenfunctions to use
- works for equation with source \( \partial_t u = \kappa \partial_{xx} u + Q(x, t) \)
- solve \( a_n(t), b_n(t) \) from the PDE + IC
3.4.9 Consider the heat equation with a known source \( q(x, t) \):

\[
\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + q(x, t) \quad \text{with} \quad u(0, t) = 0 \quad \text{and} \quad u(L, t) = 0.
\]

Assume that \( q(x, t) \) (for each \( t > 0 \)) is a piecewise smooth function of \( x \). Also assume that \( u \) and \( \partial u / \partial x \) are continuous functions of \( x \) (for \( t > 0 \)) and \( \partial^2 u / \partial x^2 \) and \( \partial u / \partial t \) are piecewise smooth. Thus,

\[
 u(x, t) = \sum_{n=1}^{\infty} b_n(t) \sin \frac{n \pi x}{L}.
\]

What ordinary differential equation does \( b_n(t) \) satisfy? Do not solve this differential equation.
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