Chapter 5: Sturm-Liouville Eigenvalue Problem

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Solution to the IBVP?

\[ c(x) \rho(x) \partial_t u = K_0 \partial_{xx} u + Q(x, t), \quad \text{with} \ x \in (0, L), \ t \geq 0 \]

IC: \( u(x, 0) = f(x), \)

BC: \( u(0, t) = \phi(t), u(L, t) = \psi(t) \)

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Extra: Numerical solution to SLEP
Recall that for Heat Equation and Wave Equation,

**HE** \( \partial_t u = \partial_{xx} u \)

**WE** \( \partial_{tt} u = \partial_{xx} u \)

**BC** \( u(a, t), \partial_x u(a, t), /mixed \)

\((a = 0, L, -L)\)

\(\phi''(x) = -\lambda \phi\)

\(\lambda_n = \left( \frac{n\pi}{L} \right)^2, \phi_n(x) = \sin \sqrt{\lambda_n} x, \cos \sqrt{\lambda_n} x, \) or both

**Quiz:** which did we use in eigenfunction expansion method?

A. \(\infty\) many eigenvalues \(\lambda_n \in \mathbb{R}\)

B. \(\{\phi_n\}\) are orthogonal

C. \(\{\phi_n\}\) are complete (Fourier Theorem)

D. TBTD conditions on \(u\)
**Review: Eigenvalue problems in PDE**

Apply it for non-constant coefficient equations?

**Heat Flow in a non-uniform rod**

\[ c(x)\rho(x)\partial_t u = \partial_x(K_0\partial_x u) + \alpha u \]

**symmetry heat flow**

\[ \partial_t u = k\frac{1}{r}\frac{\partial}{\partial r}(r\partial_r u) \]

Separation of variables \(\rightarrow\) eigenvalue problems

\[
\begin{align*}
 u(x, t) &= h(t)\phi(x) \quad \rightarrow \quad (K_0\phi')' + \alpha\phi = -\lambda c(x)\rho(x)\phi \\
u(r, t) &= h(t)\phi(r) \quad \rightarrow \quad (r\phi')' = -\lambda\phi
\end{align*}
\]

\(\rightarrow\) **Sturm-Liouville Eigenvalue Problems (SLEP)**

\[
\begin{align*}
(p(x)\phi')' + q(x)\phi &= -\lambda\sigma\phi \\
\beta_1\phi(a) + \beta_2\phi'(a) &= 0;
\beta_3\phi(b) + \beta_4\phi'(b) &= 0; \text{ other BC}
\end{align*}
\]

When does this SLEP has eigenfunctions orthogonal and complete?

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Regular SLEP:

\[(p(x)\phi')' + q(x)\phi = -\lambda\sigma\phi\]
\[\beta_1 \phi(a) + \beta_2 \phi'(a) = 0;\]
\[\beta_3 \phi(b) + \beta_4 \phi'(b) = 0;\]
\[\beta_1 + \beta_2 > 0, \beta_3 + \beta_4 > 0,\]
\[p'(x), q, \sigma \in C[a, b], p(x) > 0, \sigma(x) > 0, \forall x \in [a, b]\]

Theorem (Sturm-Liouville Theorems)

A regular SLEP has eigenvalues and eigenfunctions \[\{(\lambda_n, \phi_n)\}\] s.t.

1-2 \[\{\lambda_n\}_{n=1}^\infty\] are real and strictly increasing to \(\infty\)

3 \(\phi_n\) is the unique (up to a *factor) solution to \(\lambda_n; \phi_n\) has \(n - 1\) zeros

4 \[\{\phi_n\}_{n=1}^\infty\] is complete. That is, any piecewise smooth \(f\) can be represented by a generalized Fourier series \(f(x) \sim \sum_{n=1}^\infty a_n \phi_n(x) = \frac{1}{2}[f(x-) + f(x+)\]

5 \[\{\phi_n\}_{n=1}^\infty\] are orthogonal: \[\langle \phi_n, \phi_m \rangle_\sigma = 0\] if \(n \neq m; \langle \phi_n, \phi_n \rangle_\sigma > 0\]

6 Rayleigh quotient \(\lambda_n = -\frac{\langle L \phi_n, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle_\sigma};\)

\[\langle f, g \rangle := \int_a^b f(x)g(x)dx; \quad \langle f, g \rangle_\sigma := \int_a^b f(x)g(x)\sigma(x)dx\]
Section 5.3: Sturm-Liouville Eigenvalue Problem

Regular SLEP:

\[
(p(x)\phi'')' + q(x)\phi = -\lambda \sigma \phi \\
\beta_1 \phi(a) + \beta_2 \phi'(a) = 0; \\
\beta_3 \phi(b) + \beta_4 \phi'(b) = 0;
\]

\[p', q, \sigma \in C[a, b],
\]
\[p(x) > 0, \sigma(x) > 0, \forall x \in [a, b]
\]
\[\beta_1^2 + \beta_2^2 > 0, \beta_3^2 + \beta_4^2 > 0,
\]

Suppose \( x \in (0, L) \), we have studied SLEP \( \phi'' = -\lambda \phi \):

\[p(x) \equiv 1, q(x) \equiv 0, \sigma(x) \equiv 1. \]

Are they regular?

A \( \phi(0) = \phi(L) = 0 \)

B \( \phi'(0) = \phi'(L) = 0 \)

C \( \phi(0) + \phi'(a) = 0; \phi(L) + \phi'(L) = 0 \)

D \( \phi(-L) = \phi(L), \phi'(-L) = \phi'(L) \)
Suppose $x \in (0, L)$, we have studied SLEP $\phi'' = -\lambda \phi$:

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad \phi_n(x) = \sin \sqrt{\lambda_n}x, \cos \sqrt{\lambda_n}x, \text{ or both, depending on BC.}$$

Do we have the properties in the SL theorem?

1-2 $\{\lambda_n\}_{n=1}^{\infty}$ are real and increasing to $\infty$

3 $\phi_n$ is the unique (up to a *factor) solution to $\lambda_n$; $\phi_n$ has $n - 1$ zeros

4 $\{\phi_n\}_{n=1}^{\infty}$ is complete

5 $\{\phi_n\}_{n=1}^{\infty}$ are orthogonal

6 Rayleigh quotient $\lambda_n = -\frac{\langle L\phi_n, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle}$
Example 1 Change the equation to the form of SLEP

\[ \phi''(x) + \alpha(x)\phi' + \left[\lambda\beta(x) + \gamma(x)\right]\phi = 0 \]
Example 5.3.9. Consider the BVP:

\[ x^2 \phi'' + x \phi' + \lambda \phi = 0, \quad x \in (1, b) \]
\[ \phi(1) = 0; \phi(b) = 0 \]

(a) Write the equation in the SLE form.
(b) Show that \( \lambda \geq 0 \) for all \((\lambda, \phi)\) that solves the BVP.
(c) Determine all positive eigenvalues. Is \( \lambda = 0 \) an eigenvalue?
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Heat flow in a non-uniform rod

\[ c(x) \rho(x) \partial_t u = K_0 \partial_{xx} u, \quad \text{with} \quad x \in (0, L), \quad t > 0 \]

IC: \( u(x, 0) = f(x) \),

BC: \( u(0, t) = 0, \partial_x u(L, t) = 0 \)

Find the solution (by eigenfunction expansion):

- separation of variables \( \rightarrow \) eigenvalue problem
  \[ u(x, t) = h(t) \phi(x) \rightarrow (K_0 \phi')' = -\lambda c(x) \rho(x) \phi \]
  
- Solve the IBVP.

- positive \( c, \rho, K_0 \)
- left side \( =0^\circ \), insulated right side
Heat flow in a non-uniform rod

\[ c(x) \rho(x) \partial_t u = K_0 \partial_{xx} u, \]

IC: \( u(x, 0) = f(x), \)

BC: \( u(0, t) = 0, \partial_x u(L, t) = 0 \)

Solution

\[ u(x, t) = \sum_{i=1}^{\infty} a_n \phi_n(x) e^{-\lambda_n t}. \]

What is the large time behavior? (\( \lim_{t \to \infty} u(x, t) \) ?)

- if \( \lambda_n > 0 \) for all \( n \)

- if \( \lambda_n = 0 \) for some \( n \)

- can it happen?

- if \( \lambda_n < 0 \)?
Heat flow in a non-uniform rod

\[ c(x) \rho(x) \partial_t u = K_0 \partial_{xx} u, \]

**IC:** \( u(x, 0) = f(x), \)

**BC:** \( u(0, t) = 0, \partial_x u(L, t) = 0 \)

Solution

\[ u(x, t) = \sum_{i=1}^{\infty} a_n \phi_n(x) e^{-\lambda_n t}. \]

What does the solution look like? Numerical solution represent the solution at discrete space-time grids:

\[
\begin{align*}
x & \in [0, L] \rightarrow 0 = x_0 < x_1 < \cdots < x_{d+1} = L, \quad x_i = iL/d; \\
t & \in [0, T] \rightarrow 0 = t_0 < t_1 < \cdots < t_{N+1} = T, \quad t_j = jT/N;
\end{align*}
\]

- \( \text{Find } (\lambda_n, \phi_n); \quad (K_0 \phi')' = -\lambda c(x) \rho(x) \phi \downarrow \)
- \( \text{Find } a_n \text{ from IC} \)
- \( u(x_i, t_j) \)
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Numerical solution to SLEP

\[(K_0(x)\phi')' = -\lambda c(x) \rho(x) \phi, \quad \phi(0) = \phi(L) = 0;\]

\[L\phi = \lambda \phi \text{ with BC} \quad \rightarrow \quad Ay = \lambda y\]

SLEP \quad → \quad Linear algebra eigenvalue problem

What is \(A\)? Function \(\leftrightarrow\) vector?

\[x_i = i\Delta x, \, i = 0, \ldots, d + 1. \quad \rightarrow \quad y = (\phi(x_1), \ldots, \phi(x_d))\]