Chapter 8: Non-homogeneous Equations

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Solution to the IBVP?

\[ \partial_t u = \kappa \partial_{xx} u + Q(x, t), \quad \text{with } x \in (0, L), t \geq 0 \]

\[ u(x, 0) = f(x) \]

BC: \[ u(0, t) = \phi(t), u(L, t) = \psi(t) \]

Section 8.2: Heat flow with source and non-homo BC
Section 8.3: Methods of eigenfunction expansion (homo-BC)
Section 8.4: MEE (non-homo BC): after Chp5
Section 8.5: Forced vibrating membrane and Resonance
Section 8.6: Poisson’s Equation
Outline

Section 8.2: Heat flow with source and non-homo BC

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Section 8.2: Heat flow with source and non-homo BC

1. Time-independent BC
Consider first
\[
\frac{\partial}{\partial t} u = \kappa \frac{\partial^2}{\partial x^2} u, \quad \text{with } x \in (0, L), \ t > 0 \\
\]
\[
u(x, 0) = f(x)
\]
BC: \( u(0, t) = A, u(L, t) = B \)

1> Equilibrium solu \( u_E(x) = A + \frac{x}{L}(B - A) \).
Section 8.2: Heat flow with source and non-homo BC

1. Time-independent BC
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1> Equilibrium solu \( u_E(x) = A + \frac{x}{L} (B - A) \).
2> Displacement from Equilibrium

\[ v(x, t) = u(x, t) - u_E(x) \]

\[ \partial_t v = \kappa \partial_{xx} v, \]
\[ v(x, 0) = f(x) - u_E(x) \]
\[ v(0, t) = 0, v(L, t) = 0 \]
\[ v(x, t) = \sum_{n=0}^{\infty} a_n e^{-\kappa \lambda_n t} \sin \frac{n\pi}{L} x \]
Section 8.2: Heat flow with source and non-homo BC

1. Time-independent BC
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Extension (exe): steady source

\[ \partial_t u = \kappa \partial_{xx} u + Q(x) \]
\[ u(x, 0) = f(x) \]
\[ u(0, t) = A, u(L, t) = B \]
2. Time-dependent non-homo PDE&BC

$$\partial_t u = \kappa \partial_{xx} u + Q(x, t),$$

$$u(x, 0) = f(x)$$

$$u(0, t) = A(t), u(L, t) = B(t)$$

1> Homogenization:

▶ Equilibrium solu?

▶ May NOT be able o reduce both PDE and BC to homo.
choose one: PDE or BC?
2. Time-dependent non-homo PDE&BC

\[ \partial_t u = \kappa \partial_{xx} u + Q(x, t), \]
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choose one: PDE or BC?

⇒ reference solution \( r(x, t) \) s.t.

\[ r(0, t) = A(t); r(L, t) = B(t) \]

► any \( r^{***} \)

► \( r(x, t) = A(t) + \frac{x}{L} [B(t) - A(t)]. \)
2. Time-dependent non-homo PDE&BC

\[ \partial_t u = \kappa \partial_{xx} u + Q(x, t), \]
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▶ any \( r \) ***

▶ \( r(x, t) = A(t) + \frac{x}{L} [B(t) - A(t)]. \)

2> Displacement solution

\[ \nu(x, t) = u(x, t) - r(x, t) \]

\[ \partial_t \nu = \kappa \partial_{xx} \nu + \overline{Q}(x, t), \]
\[ \nu(x, 0) = f(x) - r(x, 0) \]
\[ \nu(0, t) = 0, \nu(L, t) = 0 \]

▶ \( \overline{Q} =? \): ***

▶ can we use separation of variables?
\[ \nu(x, t) = h(t) \phi(x) \]

\( \overline{Q} \) and no POS.

▶ Fourier series

\[ \nu(x, t) = \sum_{n=0}^{\infty} a_n(t) \sin \frac{n\pi}{L} x \]
\[ \overline{Q}(x, t) = \sum_{n=0}^{\infty} q_n(t) \sin \frac{n\pi}{L} x \]

→ method of eigenfunction expansion ↓

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separation of variables: homo PDE + homo BC

generalize $\rightarrow$ non-homo PDE + homogeneous BC

Seek solution of the form

$$u(x, t) = \sum_{n=0}^{\infty} a_n(t) \cos \frac{n\pi}{L} x + b_n(t) \sin \frac{n\pi}{L} x,$$

- homo BC determines the eigenfunctions to use (sine/cosine/both, denote by $\phi_n(x)$)
- works for equation with source $\partial_t u = \kappa \partial_{xx} u + Q(x, t)$
- solve $a_n(t), b_n(t)$ from the PDE + IC (use TBTD, under conditions)
Method of eigenfunction expansion

\[ \partial_t v = \kappa \partial_{xx} v + Q(x, t), \]
\[ v(x, 0) = g(x) \]
\[ v(0, t) = 0, v(L, t) = 0 \]
\[ \phi_n(x) = \sin \frac{n \pi}{L} x \]

\[ v(x, t) = \sum_{n=0}^{\infty} b_n(t) \phi_n(x), \quad (b_n \text{ TBD} \downarrow) \]

\[ g(x) = \sum_{n=0}^{\infty} b_n(0) \phi_n(x), \quad b_n(0) =? \]

\[ Q(x, t) = \sum_{n=0}^{\infty} q_n(t) \phi_n(x) \]
Method of eigenfunction expansion

\[ \partial_t v = \kappa \partial_{xx} v + \overline{Q}(x, t), \]

\[ v(x, 0) = g(x) \]

\[ v(0, t) = 0, v(L, t) = 0 \]

\[ \phi_n(x) = \sin \frac{n\pi}{L} x \]

\[ v(x, t) = \sum_{n=0}^{\infty} b_n(t) \phi_n(x), \quad (b_n \text{ TBD} \downarrow) \]

\[ g(x) = \sum_{n=0}^{\infty} b_n(0) \phi_n(x), \quad b_n(0) = ? \]

\[ \overline{Q}(x, t) = \sum_{n=0}^{\infty} \overline{q}_n(t) \phi_n(x) \]

TBTD \( \partial_t v, \partial_{xx} v \) PS; \( v, \partial_x v \) continuous; (BC?) \( \Rightarrow \)

\[ \partial_t v = \sum_{n=0}^{\infty} b'_n(t) \phi_n(x) \]

\[ \kappa \partial_{xx} v + \overline{Q}(x, t) = \sum_{n=0}^{\infty} \left[ -\lambda_n \kappa b_n(t) + \overline{q}_n(t) \right] \phi_n(x) \]

\[ \Rightarrow b'_n + \lambda_n \kappa b_n(t) = \overline{q}_n(t) \]

\[ b_n(t) = b_n(0) e^{-\kappa \lambda_n t} + \int_0^t e^{-\kappa \lambda_n (t-s)} \overline{q}_n(s) \, ds \]

\[ \Rightarrow \text{Check: if } \overline{Q}(x, t) = 0: b_n(t) = b_n(0) e^{-\kappa \lambda_n t}. \]
Example
Find a solution of

\[
\partial_t u = \kappa \partial_{xx} u + e^{-t} \sin 3x
\]

\[
u(x, 0) = f(x)
\]

\[
u(0, t) = 0, \quad u(\pi, t) = 1
\]
Example

Find a solution of

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\[ u(x, 0) = f(x) \]

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\[ L = \pi, \lambda_n = n^2; \]

1> reference solution:

\[ r(x, t) = 0 + \frac{x}{\pi} (1 - 0) = \frac{x}{\pi} \]
Example

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\begin{align*}
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1> reference solution:

\[
r(x, t) = 0 + \frac{x}{\pi} (1 - 0) = \frac{x}{\pi}
\]

2> let \( v(x, t) = u(x, t) - r(x, t) \)

\[
\begin{align*}
    &\partial_t v = \kappa \partial_{xx} v + \overline{Q}(x, t), \\
    &v(x, 0) = f(x) - r(x, 0) \\
    &v(0, t) = 0, v(L, t) = 0
\end{align*}
\]

\[\overline{Q}(x, t) = e^{-t} \sin 3x + 0\]

\[\text{BC} \Rightarrow \phi_n(x) = \sin nx\]

Seek \( v(x, t) = \sum_{n=0}^{\infty} b_n(t) \phi_n(x) \)
Example
Find a solution of

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\[ \text{BC} \Rightarrow \phi_n(x) = \sin nx \]

Seek \( v(x, t) = \sum_{n=0}^{\infty} b_n(t) \phi_n(x) \)

\[ \text{TBD} \partial_t v, \partial_{xx} v \text{ PS; } v, \partial_x v \text{ continuous; } \Rightarrow \]
\[ b_n(t) = b_n(0) e^{-\kappa \lambda_n t} + \int_0^t e^{-\kappa \lambda_n (t-s)} q_n(s) ds \]

\[ u(x, t) = v(x, t) + \frac{x}{\pi} \]

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In the method of eigenfunction expansion, what if

- TBTD conditions not satisfied
- More general equations \( \partial_{xx} \rightarrow \frac{d}{dx}(p(x)\frac{d}{dx}) \)

Back to it after chapter 5.
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