An introduction by discussion.

Population evolution

\[
\frac{dN}{dt} = \alpha(t) N(t) \quad \text{or} \quad N(t) = K_{0} \cdot e^{tQ}
\]

\[
N(t) - N(0) = \int_{0}^{t} \alpha(s) N(s) ds
\]

randomness,

1. \[X(t) = X_{0}N(t) \quad \text{or} \quad N(t) = N_{0}X(t)\]
2. \[\dot{N}(t) = N_{0} + \int_{0}^{t} X_{0}(s) ds\]
3. \[\dot{N}(t) = N_{0} + \int_{0}^{t} X_{0}(s) ds + \int_{0}^{t} X_{0}(s) ds\]

Q1. How to solve the equation?
Q2. How to describe the randomness \((X, N)\)?
Q3. When \(Q \approx 0\)?
Q4. If observe \(Z(t) = h(N(t)) + \text{Noise}\). How to estimate \(N(t)\)?
Chapter 2: Probability space, random variables, stochastic processes.

Def 2.1.1 Probability space \((\Omega, \mathcal{F}, P)\): \(\Omega\) is a set, \(\mathcal{F}\) is a \(\sigma\)-algebra, \(P\) is a prob. meas.

1. \(P(\emptyset) = 0, P(\Omega) = 1\)
2. If \(A_1, A_2, \ldots \in \mathcal{F}\) are disjoint, then
   \[P(\bigcup A_i) = \sum_i P(A_i) \]
3. \(\emptyset \in \mathcal{F}\)
4. \(A^c \in \mathcal{F} \quad \text{ for each } A \in \mathcal{F}\) (complement)
5. \(\mathcal{F}\) is complete if \(\mathcal{F}\) contains all subsets \(G\) of \(\Omega\) w/ \(P\)-outer meas. zero.

- \(P(G) = \inf\{P(F) \mid G \subseteq F, F \in \mathcal{F}\}\)

- \(F\)-measurable sets: \(F \in \mathcal{F}\)

- \(\mathcal{U}\) = a family of subsets of \(\Omega\): \(\mathcal{H} = \cap \{H : H \subseteq \sigma\text{-algebra generated by } \mathcal{U}\}\)

Example 1. Borel \(\sigma\)-algebra \(\mathcal{B}\) for \(\Omega = \mathbb{R}^n\): generated by all open subsets.

Example 2. \(\mathcal{H}_x\) = \(\sigma\)-algebra generated by \(X : \Omega \to \mathbb{R}^n\)

\[\mathcal{H}_x = \text{the smallest } \sigma\text{-algebra on } \Omega \text{ containing all sets } X^{-1}(U), U \subseteq \mathbb{R}^n \text{ open.}\]

Random variable: \(X : \Omega \to \mathbb{R}^n\)

1. A function \(X : \Omega \to \mathbb{R}^n\) is \(F\)-measurable if \(X^{-1}(U) \in \mathcal{F}\), \(U \subseteq \mathbb{R}^n\) open.

Distribution of \(X\): \(M_X(B) = X^{-1}(B), B \subseteq \mathbb{R}^n\)

Expectation: \(1E[X] = \sum_{\omega \in \Omega} X(\omega) P(\omega) = \int_{\mathbb{R}^n} x d\mu_X(x)\)

\(L^p(\omega):\) \(1E[|X|^p] < \infty\)

\(L^p(\mathcal{F}):\) \(\{f : \mathbb{R}^n \to \mathbb{R} \text{ Borel measurable s.t. } 1E[|f(X)|^p] = \int_{\mathbb{R}^n} |f(x)|^p d\mu_X(x) < \infty\}\)

\(1E[1_A] = P(A)\)
Independence

**Lemma 2.1.2 (Doob-Dynkin lemma)** If $X, Y: \Omega \to \mathbb{R}^n$ are two given r.v.s, then $Y$ is $\mathcal{H}_X$-meas.

IFF \exists a Borel measurable function $g: \mathbb{R}^n \to \mathbb{R}^n$ s.t.: $Y = g(X)$.

**Def 2.1.3** Two sets $A, B \in \mathcal{F}$ are independent if $\mathbb{P}(A \cap B) = \mathbb{P}(A) \mathbb{P}(B)$.

Two families $H_1, H_2 \in \mathcal{F}$ are indpt if $\mathbb{P}(H_1 \cap H_2) = \mathbb{P}(H_1) \mathbb{P}(H_2)$, $\forall H_1, H_2 \in \mathcal{F}$.

A collection of families $\{H_i, i \in I\}$ are indpt if $\mathbb{P}(H_1 \cap \cdots \cap H_k) = \mathbb{P}(H_1) \cdots \mathbb{P}(H_k)$, $\forall H_1, \cdots, H_k \in \mathcal{F}$.

A collection of r.v. $\{X_i, i \in I\}$ is indpt if $\{X_i, i \in I\}$ is indpt.

$\Rightarrow$ If $X_1, X_2$ are indpt, then $\mathbb{E}[X_1X_2] = \mathbb{E}[X_1] \mathbb{E}[X_2]$, if $\mathbb{E}[X_1] < \infty$, $\mathbb{E}[X_2] < \infty$.

**Conditional expectation** (later)
§ Chp 2: Mathematical Preliminaries & Notations

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space.

**Definition 2.14** A stochastic process is a parametrized collection of random variables \(\{X_t\}_{t \in T}\) defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). \([\mathbb{R}^n\text{-valued}].\)

- Parameter space \(T: [0, \infty), \mathbb{R}, H;\) set
- \(\mathcal{A}\) a \(\sigma\)-algebra, a prob.

**Random variable view:** \(t \mapsto X_t: \Omega \rightarrow \mathbb{R}^n\) a random variable

**Function view:** \(X: T \times \Omega \rightarrow \mathbb{R}^n\) (jointly measurable)

**Path view:** \(\omega \mapsto X(\omega): t \mapsto \mathbb{R}^n\) a path/trajectory.

\[\text{Identify } \omega \text{ with the path } X(\omega) \Rightarrow \mathcal{F} \subseteq (\mathbb{R}^n)^T = C(T, \mathbb{R}^n)\]

Measure on path space: \((\mathcal{F}, \mathcal{B}, \mathbb{P}) \prec \mathcal{B} = \text{Borel } \sigma\text{-algebra on } \mathcal{F}\).

\(\{\omega = X(\omega): w(t) \in F_t, \cdots, w(t_k) \in F_k\} \text{ with } F_t \in \mathbb{R}^n \text{ Borel sets}.\)

Q1: How do we "specify" an SP?

Finite-dimensional distributions of \(\{X_t\}_{t \in T}\) are the dests. of \((X_t, \cdots, X_k), \forall k \in T\).

i.e. the measures \(\mu_{t_0\cdots t_k}\) defined on \(\mathbb{R}^{nk}\) by

\[\mu_{t_0\cdots t_k}(F_{t_0} \times F_{t_1} \times \cdots \times F_{t_k}) = \mathbb{P}(X_{t_0} \in F_{t_0}, \cdots, X_{t_k} \in F_{t_k}), \quad t \in T\]

where \(\{F_{t_i}\}\) are Borel sets in \(\mathbb{R}^n\).

From experiments or numerical simulations we have information only about the \(S\).

Q2. Given a family of FDDs, can we construct an SP? Yes \& Kolmogorov Thm 2.1.5 (Kolmogorov's extension theorem)

(consistency conditions required)

For all \(t_0 \cdots t_k \in T, k \in \mathbb{N}\), let \(\nu_{t_0 \cdots t_k}\) be prob. measures on \(\mathbb{R}^{n_k}\) sq.

i) permutation consistent: \(\nu_{t_0 \cdots t_k}(F_{t_0} \times \cdots \times F_{t_k}) = \nu_{t_0 \cdots t_k}(F_{t_0} \cdots F_{t_k}), \forall \sigma;\)

ii) marginal consistent: \(\nu_{t_0 \cdots t_k}(F_{t_0} \times \cdots \times F_{t_k}) = \nu_{t_0 \cdots t_{k-1}}(F_{t_0} \times \cdots \times F_{t_{k-1}} \times \mathbb{R}^{n_k} \times \cdots \times \mathbb{R}^n), \forall k \geq 1\)

Then \(\exists\) a prob. space \((\Omega, \mathcal{F}, \mathbb{P})\) and an SP \(\{X_t\}_t\) s.t. \(\nu_{t_0 \cdots t_k} \sim X_{t_0 \cdots t_k}.\)
Q2: Can the FDDs determine an SP uniquely? NO for general SP.

Yes for cts SP. 

(Version & cts.) 

Def. 2.2.2: Sps that $X^{w}$ & $Y^{w}$ are SP on $(\Omega, F, P)$. We say that $Y^{w}$ is a version of (or a modification of) $X^{w}$ if

$$P(\{w: X^{w}(t) = Y^{w}(t)\}) = 1 \quad \text{for all } t.$$ 

Note: $X$ & $Y$ have the same FDDs ⇒ the two Sps are the same, but path properties may be different.

Exe 2.9: Let $(\Omega, F, P) = (\mathbb{R}, \mathcal{B}, \mu)$ Borel σ-algebra, $\mu$: No mass on single pt.

Define $X^{w}(t) = \begin{cases} 1 & \text{if } t = w \\ 0 & \text{otherwise, } \end{cases}$ $Y^{w}(t) \equiv 0$, $\forall t, w$.

Then: $P(X = Y) = P(X = 0) = P(w: t = w) = 1$ discontinuous for each trajectory.

Thm 2.2.3: (Kolmogorov's continuity theorem)

Sps that the process $X = \{X^{w}\}_{w \in \Omega}$ satisfies the moment condition, $\forall T, a, \beta > 0$.

$$|E[|X - X_s|^{\beta}]| \leq D |t - s|^{1 + \beta}, \quad 0 \leq s, t \leq T.$$ 

Then there exists a cts version of $X$.

[Proof: see eg. Nualart Appendix.]
Typical SP:

1. **Gaussian Processes (GP).**

   **Def.:** A dts-time GP is an SP whose FDDs are Gaussian, i.e.,
   \[ X_{t_1:t_k} \sim N(\mathbf{m}_k, \mathbf{C}_k) \Leftrightarrow \exists \mathbf{m}_k, \mathbf{C}_k > 0 \text{ s.t. } \mathbb{E}[e^{i\mathbf{y}^\top \mathbf{x}}] = e^{\frac{1}{2} \mathbf{y}^\top \mathbf{C}_k \mathbf{y}} \]
   \[ \mathbf{R}^k \times \mathbf{R}^k \]

   **Fact:** A GP is completely characterized by its mean \( \mathbf{m}(t) = \mathbb{E}[X_t] \) & covariance \( \mathbf{C}(s,t) = \mathbb{E}[(X_s - \mathbf{m}(s))(X_t - \mathbf{m}(t))] \).

   Numerical simulation of GP (law-D):
   \[ t_1, \ldots, t_n; \quad \mathbf{m}(t) \to \mathbf{m}_n \in \mathbb{R}^n, \]
   \[ X_{t_1:t_n} = \mathbf{m}_n + \sum_{i=1}^n N(0, I_d) \; ; \quad \mathbf{C}(s,t) \to \mathbf{C}_n \in \mathbb{R}^{n \times n} \]

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**Example:** Brownian motion

Robert Brown (1828): pollen grains in water; irregular motion

: average of random collisions with the water molecular.

Bachelier (1900): quantitative work; stock price fluctuations

Einstein (1905): derive the transition density from molecular-kinetic theory

Wiener (1923, 24): rigorous math, \( \mathbb{E} \) interpolation & path of heat

Levy (1939, 1948): construction of \( B_t \); passage time & related functionals

Kolmogorov (1933) construction: FDDs

Dantelli (1918/19) integral on a space of sequence

**Definition (Bm).** A Bm \( \{B_t\}_{t \geq 0} \) starting from \( B_0 = x \) is an SP whose FDDs satisfy:

\[
I^x(B_t \in F_1, \ldots, B_k \in F_k) = \int_{F_1 \times F_2 \times \ldots \times F_k} \prod_{i=1}^k p(t_i, x_i, y_i) \, dy_1 \ldots dy_k
\]

\( \forall \ 0 < t_1 < \ldots < t_k \) and \( F_1, \ldots, F_k \) being Borel sets on \( \mathbb{R}^k \),

where \( p(t,x,y) = \left(2\pi t\right)^{-k/2} e^{-\frac{1}{2} (y-x)^2/t}, \quad x, y \in \mathbb{R}, \ t > 0. \)

**Basic properties:**

1. Bm is a GP.
2. Indpt increments.
3. Has a cts version.

**Remk1.** Bm defined by FDDs is NOT unique; choice of version via cts paths.

\[
\mathbb{E}[B_t - B_s] = \frac{1}{2}(t-s)
\]

Kolmogorov's continuity theorem

Path space view: Bm is just the space \( C([0,\infty) , \mathbb{R}) \) equipped w/ a mean \( I^x \):

\[
\rightarrow (\mathcal{D}, B, I^x) \text{ canonical Bm.}
\]

**Remk2.** Other definitions:

1. Bm is a GP w/ cts sample paths & covariance centered.
2. See KS 91 for a definition \( \int_t f_t \).
3. Levy characterization:
   \( X_t \) is a martingale (w.r.t. its own filtration)
   \( X_{t_1}, X_{t_1}, \ldots, X_t \) is a mG. \( = \mathcal{M}^{-} (X_t) \).

**Remk3.** Construction of Bm.

1. Rescaled random walk. Let \( \{X_t \} \) r.v.d. mean 0 variance 1.
   \[
   S_n = \sum_{i=1}^{n} X_i , \ n \geq 0.
   \]
   Define
   \[
   W_t = \sqrt{n} S_{\lfloor nt \rfloor} + (nt - \lfloor nt \rfloor)^{1/2} X_{\lfloor nt \rfloor + 1}
   \]
2. Levy's construction based on interpolation (no path).
3. Payley - Wiener
   \[
   B_t = \frac{d}{dv} \left( \frac{v}{\sqrt{2\pi n}} \right)^{1/2} \tilde{S}_n , \quad \tilde{S}_n \sim \mathcal{N}(0,1) \text{ i.i.d.}
   \]
**Other properties**: (Prop 1.5, Part 4)

- Scaling: $B_t := c^{-rac{1}{2}}B_{ct}$ is also a BM  \( (\text{E12.1.6}) \)
- Shifting: \( (B_{t+\tau} - B_{\tau}) \) is a BM, $\forall \tau > 0$.  \( (\text{E12.1.2}) \)
- Time reversal: \( (B_{t-\tau} - B_\tau)_{t \in [0,1]} \) is a BM.
- Inversion: \( (X_0 = 0, X_t = tW_t)_{t \geq 0} \) is a BM.

**Example 2**: $fB_m, B^m_t$ is a GP with its sample paths whose covariance is given by

\[
\mathbb{E} [B^m_tB^m_s] = \frac{1}{2} (s^{2H} + t^{2H} - Ht^ss^{1-2H}), \quad H \in (0,1)
\]

When $H = \frac{1}{2}$, we have BM.

**Example 3**: Brownian Bridge \( W_t = B_t - tB_1, \quad t \in [0,1] \)

**Example 4**: Ornstein-Uhlenbeck process.

\[
X_t = e^{-\theta t}X_0 + \int_0^t e^{-\theta(t-s)} dW_s
\]

- A GP
- \( \mathbb{E}[X_t] = X_0 e^{-\theta t} \)
- \( \text{Cov}(X_t, X_0) = \frac{1}{2} (1 - e^{-2\theta s}) e^{-\theta(t-s)}, \quad t > s \)

17. Compute (b) from (a):
- \( \mathbb{V}(X_t) = \int_0^t e^{-2\theta(t-s)} ds = \frac{1}{2\theta} (1 - e^{-2\theta t}) \)
- \( \mathbb{V} X_t^2 = e^{-2\theta t} \mathbb{E} X_t^2 + \frac{1}{2\theta} (1 - e^{-2\theta t}) \)

\[
\text{Cov}(X_t, X_0) = \mathbb{E} [ (X_t - \mathbb{E} X_t)(X_0 - \mathbb{E} X_0) ] = \mathbb{E} [X_t X_0] - \mathbb{E} X_t \mathbb{E} X_0 = \frac{1}{2\theta} (1 - e^{-2\theta s}) e^{-\theta(t-s)}
\]

\[
X_t = e^{-\theta t}X_0 + \int_0^t e^{-\theta(t-s)} dB_s
\]

\[
\Rightarrow \mathbb{E} [X_t X_0] = e^{-\theta(t-s)} \left( t e^{-\theta s} \right) + 0 = e^{-\theta(t-s)} \left( e^{-2\theta s} X_0^2 + \frac{1}{2\theta} (1 - e^{-2\theta s}) \right)
\]

\[
\mathbb{V} [X_t | X_0] = X_0 e^{-\theta t} - e^{-2\theta s} = \frac{1}{2} e^{-\theta(t-s)}
\]

17. stationary, $t \rightarrow 0$: \( \lim_{t \rightarrow 0} \mathbb{E}[X_t] = 0 \quad \Rightarrow \text{stationary, } X_t \sim N(0, \frac{1}{2\theta}) \)

\( \lim_{t \rightarrow 0} \mathbb{V}[X_t] = \frac{1}{2\theta} \)

\[
\text{Cov}(X_t, X_0) = \text{same as above}
\]

\( \text{(Dynamic properties) } \)
2. Stationary Processes.

Definition (Strong stationary process) An SP is strongly stationary if all FPPs are invariant under time translation: \( X_{t_1, t_2} \sim X_{t_2, t_1}, \forall \ t_1, t_2, k. \)

Example 1. Let \( Z \) be a r.v. & let \( X_n \equiv Z, \forall n. \) Then \( \{X_n\} \) is stationary

Example 2. \( \alpha \& \beta \) seq.

Example 3. OU \( w/ \) \( L_\text{C} \) being the stationary distribution: \( \mathbb{E} \infty \int_{0}^{\infty} \mathbb{E}(X_0, X_\tau) = \frac{1}{2b} (1 - e^{-\alpha \tau}) e^{-\beta \tau}. \)

Def. (Weak stationary/2nd-order stationary) if \( \mathbb{E} X_t = \mu \)
\[
\mathbb{E}[X_t - \mu] [X_s - \mu] = C(t - s)
\]

Prop 1.3 (Fact) (Ergodicity of stationary process) \( \{X_t\}\) is weak stationary, \( \mu \in L. \) Assume \( \mathbb{E} X_t \in L^2(0, \infty). \) Then
\[
\lim_{T \to \infty} \mathbb{E} \left[ \left| \int_0^T X_s \, ds - \mu \right|^2 \right] = 0.
\]


\{ \( X_t, t \in [0, 1] \) \} be an \( L^2 \) process w/ a mean \& corr. \( R(t, s) \)

Let \( \lambda_n, e_n(t) \) be the eigen-pairs of \( R \)-integral operator on \( L^2[0, 1]. \) Then
\[
X_t = \sum_{n=1}^{\infty} \lambda_n e_n(t), \quad t \in [0, 1], \quad \lambda_n = \int_0^1 X_t e_n(t) \, dt
\]

converges in \( L^2 \) to \( X_t \), uniformly in \( t. \)
\[
\|E_n\| = 0; \quad \|E[X_n]\| = \lambda_n \|e_n\|.
\]
2.8. Let $B_t$ be Brownian motion on $[0, T]$, $B_0 = 0$. Put $E = E^0$.

(a) Use $E^0 \{ e^{\beta B_t} \} = e^{-\frac{\beta^2}{2} t} \frac{e^{\beta B_0}}{\sqrt{2\pi t}}$ for all $t \geq 0$ and $\beta > 0$ to prove that

$$E e^{iuB_t} = e^{-\frac{u^2}{2} t}, \quad \forall u \in \mathbb{R}$$

(b) Use the power series expansion of the exponential function on both sides, compare the terms w the same power of $u$ and deduce that

$$E[B^2_t] = 2t$$
$$E[B^4_t] = \frac{24}{5} t^2$$
$$E[B^6_t] = \frac{168}{7} t^3$$
$$E[B^8_t] = \frac{128}{35} t^4$$

(c) Alternative of (b). Prove that (2.2.2):

$$E[(B_t, B_{2t}, \ldots, B_{kt})] = \sqrt{t} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^k} f(x) e^{-\frac{x^2}{2t}} dx$$

implies that

$$E[f(B_t)] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-\frac{x^2}{2t}} dx$$

for all functions $f$ s.t. the integrals on the right exists. Then apply this to $f(x) = x^k$ and use induction on $k$.

(d) Prove

$$E[|B_t - B_s|^4] = n(n+2)(t-s)^2$$

by using induction.
2.8. (a),(b),(c), follow the direction.

(d). \[ n=1, \quad IE[B_t^4] = 3t^2 = n(n+2)t^2 \]
\[ n=2 \quad IE[|B_t B_{t+2}|^4] = IE[|B_t^2 + B_{t+2}^2|^2] = IE[|B_t^2 + B_{t+2} - B_{t+2} B_{t+2}|^2] \]
\[ = 3t^2 + 3t^2 + 2t^2 = 8t^2 = n(n+2)t^2 \]
\[ n \geq 2: \quad IE[|B_t, ..., B_n|^4] = IE[|B_t^2 + \sum_{k=t+2}^n B_k^2|^2] \]
\[ = IE[B_t^2 + \sum_{k=t+2}^n B_k^2 + \sum_{k=t+2}^n \sum_{j=k+1}^n B_k^2 B_j^2] \]
\[ = 3t^2 + 3t(n) + IE(\sum_{k=t+2}^n B_k^2)^2 \]
\[ = \left(3 + 2(n-1) + n^2\right)t^2 = \left(n^2 + 2n\right)t^2. \]

2.16. By GP definition, verify the covariance.

\[ \frac{\Xi_t (B_{t+k})^2}{\Xi_t (B_t)^2} \rightarrow \frac{\Delta = \max_{k \in \mathbb{N}} \Delta_k}{\Delta_t} \]

2.17. Show that \( B_t \) has unbounded IV a.s. from 
\[ IE\left[\sum_{t_k\in\mathbb{N}} |B_{t_k}|^2 \right] = 2 \sum_{t_k \in \mathbb{N}} \Delta_k^2 \rightarrow 0 \Rightarrow V_t^2 = \sum_{t_k \in \mathbb{N}} |B_{t_k}|^2 \rightarrow t \text{ a.s.} \]

**Proof:**

1. Note that (a) implies 
\[ Y_t^2 = \sum_{t_k \in \mathbb{N}} |B_{t_k}|^2 \rightarrow t \text{ a.s.} \]

(You can also use \( B_t^2 \rightarrow t \text{ a.s. here.} \))

2. Let 
\[ V_t(w) = \frac{\sum_{t_k \in \mathbb{N}} |B_{t_k}|^2}{\Delta_t^2} \]
\[ \text{Then, noting that } \sup_{\mathbb{R}^2} \frac{\Delta}{\Delta_0} \rightarrow 0 \text{ and } B_t(w) \text{ is ctf,} \]
\[ \text{we have, as } \Delta \rightarrow 0, \]
\[ Y_t^2 = \sum_{t_k \in \mathbb{N}} |B_{t_k}(w)|^2 \rightarrow \sup_{t_k \in \mathbb{N}} |B_{t_k}|^2 \rightarrow 0 \rightarrow V_t(w) \]

\[ \Rightarrow \quad IP(w, V_t(w) < 10) = 0. \]