Dirichlet Problems and Electrical Networks

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Overview

1. Motivation
   - Discrete-time Markov Chains
   - Pólya's Theorem
2. Dirichlet Problems
   - Harmonic Functions
   - Reversibility
3. Electrical Networks
   - Finite Networks
   - Infinite Networks
4. Applications
   - Back to Pólya's Theorem
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Basic Settings

The following are some notations that we use throughout these slides:

$$\mathbb{N} = \{1, 2, 3, \ldots\};$$
$$\mathbb{N}_0 = \{0, 1, 2, \ldots\}.$$

- The triple $(\Omega, \mathscr{F}, \mathbb{P})$ is a probability space, all the random variables of our interests are defined on it.
- The state space S is countable. It is equipped with discrete topology and $\mathscr{S}$ is the Borel σ-algebra on S.

Basic Settings

Let $X = (X_n)_{n\in\mathbb{N}_0}$ be a time-homogeneous Markov chain adapted to $(\mathscr{F}_n^X)_{n\in\mathbb{N}_0}$ taking values in S. Its corresponding transition probabilities are specified by a function $P(\cdot,\cdot) : S \times S \to [0,1]$, in other words,

$$P(x,y) = \mathbb{P}(X_{n+1} = y \,|\, X_n = x) \text{ for all } n \in \mathbb{N}_0.$$

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Conventions

If the distribution of X_0 under $\mathbb{P}$ is μ, that is,

$$\mathbb{P}(X_0 \in A) = \mu(A) \text{ for all } A \in \mathscr{S},$$

then we write $\mathbb{P}$ as $\mathbb{P}_\mu$to indicate the initial distribution. If $\mu = \delta_x$, we use $\mathbb{P}_x$ as an abbreviation for $\mathbb{P}_{\delta_x}$.

Simple Symmetric Random Walk on $\mathbb{Z}^d$

Given $d \in \mathbb{N}$.

Definition

A simple symmetric random walk X on $\mathbb{Z}^d$ is a Markov chain whose state space S is $\mathbb{Z}^d$ with transition probability P given by

$$P(x, y) = \begin{cases} \frac{1}{2d} & \text{if } |x - y| = 1; \\ 0 & \text{otherwise.} \end{cases}$$

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Q: Is this chain recurrent or transient?

Pólya's Theorem

Pólya's Theorem (1921)

Simple symmetric random walk on $\mathbb{Z}^d$ is recurrent if and only if $d \leq 2$.

As a quote from Kakutani put: "A drunken man will eventually find his way home but a drunken bird may get lost forever."

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Many traditional proofs are known:

- Combinatorial methods;
- Chung-Fuchs theorem via Fourier-Stieltjes transform;
- Local central limit theorem.
- See more in Durrett [2019]

Pólya's Theorem

However, all of the aforementioned methods are not very robust and would need a substantial improvement in order to cope with even a very small local change in P.

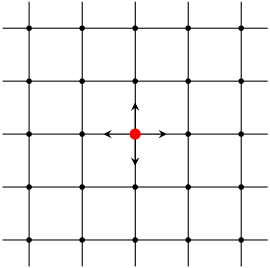
They all exploit the independence and identical distribution structure in the construction of $X = (X_n)_{n\in\mathbb{N}_0}$ as

$$X_n := X_0 + \sum_{k=1}^{n} \xi_k,$$

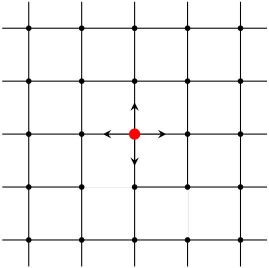
where ξ_k's are i.i.d. with distribution

$$\xi_k = \begin{cases} e_i & \text{for each } i = 1,\dots,d \text{ with probability } \frac{1}{2d}; \\ -e_i & \text{for each } i = 1,\dots,d \text{ with probability } \frac{1}{2d}; \\ 0 & \text{otherwise.} \end{cases}$$

Pólya's Theorem



Integer lattice $\mathbb{Z}^2$

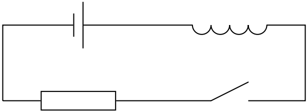


$\mathbb{Z}^2$ with removed edges

Q: Is this chain still recurrent if some edges are removed from the integer lattice $\mathbb{Z}^2$?

Pólya's Theorem

Intuitively, such a local change should not affect the global behavior such as recurrence if irreducibility still holds. We need to resort to other tools to answer this question rigorously.



Electrical Networks!

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Harmonic Functions

Definition

Let A be a nonempty subset of S. A function $f : S \to \mathbb{R}$ is called harmonic on A if

$$Pf(x) := \sum_{y \in S} f(y)P(x,y) \tag{2.1}$$

exists for all $x \in A$ and $Pf = f$ on A.

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Example

- If X is irreducible and transient, given any $a \in S$, the Green function $G(x,a) := \mathbb{E}_x[\sum_{n=0}^{\infty} \mathbb{1}_{\{X_n = a\}}] = \sum_{n=0}^{\infty} P^{(n)}(x,a)$ is harmonic on $S \setminus \{a\}$.

Harmonic functions

Example

- Let B be an nonempty proper subset of S and $\tau_B^+ = \inf\{n \in \mathbb{N} : X_n \in B\}$. Let $g : B \to \mathbb{R}$ be a bounded function. Then the function

$$f(x) := \begin{cases} \mathbb{E}_x\left[g\left(X_{\tau_B^+}\right)\mathbb{1}_{\{\tau_B^+<\infty\}}\right] & \text{if } x \in S \setminus B; \\ g(x) & \text{if } x \in B \end{cases} \tag{2.2}$$

is harmonic on $S \setminus B$.

Example 4 can be verified by an easy first-step analysis. It provides us with a solution to the Dirichlet problem in the next slide.

Dirichlet Problem

Definition

Let B be a nonempty proper subset of S. The Dirichlet problem is defined by finding $f : S \to \mathbb{R}$ such that

$$\begin{cases} (P - I)f(x) = 0 & \text{for } x \in S \setminus B; \\ f(x) = g(x) & \text{for } x \in B, \end{cases} \tag{2.3}$$

where $g : B \to \mathbb{R}$ is a given bounded function.

We have shown the existence of the solution to (2.3). Clearly, the solution is not unique in general, for example, when $P = I$, then any function that coincides with g on B is a solution.

Absorbing Chain

In order to describe formally the irreducibility condition that we have to impose, we introduce the transition probability P_B of the chain $X^B = (X_n^B)_{n\in\mathbb{N}_0}$ absorbed upon reaching B by

$$P_B(x,y) := \begin{cases} P(x,y) & \text{if } x \in S \setminus B; \\ \delta_{xy} & \text{if } x \in B. \end{cases}$$

For $x, y \in S$, define the hitting probability

$$\rho_B(x,y) := \mathbb{P}_x(\tau_y^+ < \infty),$$

where τ_y^+ is the hitting time $\inf\{n \in \mathbb{N} : X_n^B = y\}$. Let

$$S_B(x) := \{y \in S : \rho_B(x,y) > 0\}.$$

Maximum principle

Theorem (Maximum principle)

Let f be a harmonic function on $S \setminus B$.

1. *If there exists an $x_0 \in S \setminus B$ such that*
$$f(x_0) = \sup_{x \in S_B(x_0)} f(x),$$
then $f(y) = f(x_0)$ for any $y \in S_B(x_0)$.
2. *In particular, if $\rho_B(x,y) > 0$ for all $x, y \in S \setminus B$, and if there exists an $x_0 \in S \setminus B$ such that $f(x_0) = \sup(f(S))$, then $f(y) = f(x_0)$ for any $y \in S \setminus B$.*

Uniqueness of the Solution

Superposition principle

If f and g are harmonic functions on A and let $\alpha, \beta \in \mathbb{R}$. Then $\alpha f + \beta g$ is also harmonic on A.

Theorem (Uniqueness)

Let $X = (X_n)_{n \in \mathbb{N}_0}$ be an irreducible Markov chain with transition probability P. Let $B \subsetneq S$ be a nonempty proper subset of S with $|S \setminus B| < \infty$. Given two harmonic functions f_1 and f_2 on $S \setminus B$, if $f_1 = f_2$ on B, then $f_1 = f_2$ on S. In other words, the Dirichlet problem (2.3) *has a unique solution.*

The proof is based on maximum principle and superposition principle for Harmonic functions.

Random Walks on Graphs

Recall a measure μ on S is called reversible if it satisfies the detailed balanced equation

$$\mu(x)P(x,y) = \mu(y)P(y,x).$$

If μ is a probability measure, then the corresponding Markov chain is called reversible.

Example

Let $|S| < \infty$, given the adjacency matrix $A = (C(x,y))_{x,y\in S}$ for a weighted graph without self-loops and isolated vertices, one can construct the transition probabilities P as

$$C(x) := \sum_{y\in S} C(x,y), \quad P(x,y) = \frac{C(x,y)}{C(x)}. \tag{2.4}$$

Graph Laplacian

For a function $u : S \to \mathbb{R}$, consider the function

$$Lu(x) = \mathbb{E}_x[u(X_1) - u(X_0)] = (P - I)u(x),$$

it measures the expected change in one step of the chain if the chain starts at state x. Let D be the degree matrix of the graph, then

$$L = P - I = D^{-1}(A - D)$$

is called the weighted graph Laplacian.

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Example (Simple symmetric random walk on $\mathbb{Z}^d$)

$$Lu(x) = \frac{1}{2d}\sum_{i=1}^{d}(u(x+e_i) - u(x) - (u(x) - u(x-e_i))).$$

$$L \approx \frac{1}{2d}\Delta$$

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Electrical Circuits

A finite electrical network is a undirected connected simple graph G with vertex set S and edge set E on which a weight function C is defined such that $|S| < \infty$.

Definition

The conductance of an edge $e \in E$ with $e = \{x, y\}$ is

$$C : E \to [0, \infty), \quad C : e \mapsto C(e) = C(\{x, y\}) =: C(x, y) = C(y, x).$$

The reciprocal $R(e) := 1/C(e)$ is called the resistance of the edge e.

An electrical network is denoted by the pair (G, C) . For $x, y \in S$, we write $x \sim y$ to indicate that $\{x, y\} \in E$.

Electrical Circuits

Let B be the boundary set in the Dirichlet problem (2.3).

Definition

A function θ on ordered pairs of adjacent vertices in S is called a flow if

$$\theta(x,y) = -\theta(y,x) \text{ for all } x \sim y \text{ in } S,$$
$$\sum_{y \sim x} \theta(x,y) = 0 \text{ for all } x \in S \setminus B.$$

The second equation satisfied by θ is called Kirchhoff's node law.

Electrical Circuits

Definition

A flow I is called a current flow if there exists a function $u : S \to \mathbb{R}$ such that Ohm's law is satisfied:

$$I(x,y) = \frac{u(x) - u(y)}{R(x,y)} \text{ for all } x, y \in S \text{ and } x \neq y.$$

In this case, $I(x,y)$ is called the current flow from x to y and $u(x)$ is called the electrical potential or voltage at x.

In physics, Ohm's law is an empirical statement about electrical current's linear response to voltage differences.

Voltage Function is Harmonic

Theorem

An voltage u on a network (G, C) is a harmonic function on $S \setminus B$. If its corresponding reversible Markov chain is irreducible, then u is uniquely determined by its values on B.

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Proof.

By Kirchhoff's node law and Ohm's law, given $x \in S \setminus B$,

$$0 = \sum_{y \sim x} I(x,y) = \sum_{y \sim x} C(x,y)(u(x) - u(y))$$
$$= C(x) \sum_{y \sim x} P(x,y)(u(x) - u(y))$$
$$= -C(x)Lu(x).$$

□

Effective Conductance and Resistance

Suppose the boundary $B \subseteq S$ can be further partitioned into two disjoint nonempty set A and Z such that $B = A \uplus Z$.

Unit Voltage

The set A is the set of sources and Z is the set of sinks. The boundary condition on B is $g = \mathbb{1}_A$.

Definition

The total current between A and Z is defined by

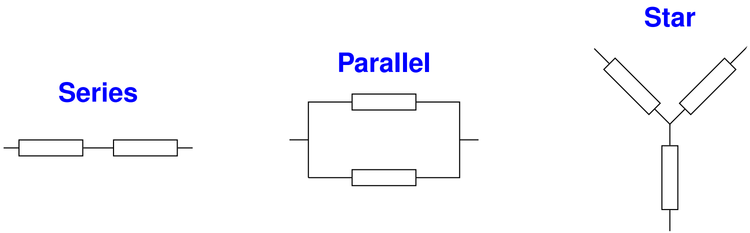
$$I(A \to Z) := \sum_{x \in A} \sum_{y \sim x} I(x, y).$$

The effective resistance is $R_{eff}(A \to Z) = \frac{1}{I(A \to Z)}$, its reciprocal $C_{eff}(A \to Z)$ is called the effective conductance.

Effective Conductance and Resistance

It is easy to verify that the effective resistance is the same if A and Z are switched, so it is usually denoted by $R_{\mathit{eff}}(A \leftrightarrow Z)$.

Our main goal is to compute the effective conductance between two sets of vertices. The idea is to reduce a complex network to a combination of the following three types of connections.



Network Reduction

Not every network can be reduced to a combination of the previous three cases. Sometimes it might be quite tricky, many algorithms are developed to reduce a large-scale network. Let's content ourselves with this brief discussion and move to the topic why we care about effective conductance probabilistically.

Probabilistic interpretation

For simplicity, assume $A = \{a\}$ is a singleton, the following theorem demonstrates the connection between certain probabilistic quantity and effective conductance.

Theorem

Given an irreducible reversible Markov chain X, for $a \in S$ and nonempty $Z \subseteq S$ with $a \notin Z$,

$$\mathbb{P}_a(\tau_Z < \tau_a^+)C(a) = C(a \leftrightarrow Z).$$

The proof is based on the uniqueness and existence of the solution to the Dirichlet problem (2.3) with $B = \{a\} \uplus Z$.

Question

We have only considered finite electrical networks in our discussion so far, but $\mathbb{Z}^d$ is an infinite network. How do we go from the case when $|S| < \infty$ to the one when $|S| = \infty$?

Pass to the Limit!

Approximating an Infinite Network

Given an infinite electrical network (G, C). Let $(G_n)_{n\in\mathbb{N}}$ be an increasing sequence of finite subgraphs of G such that $\bigcup_{n=1}^{\infty} G_n = G$. Let Z_n be the set of vertices in $G \setminus G_n$. Let G_n^W be the graph obtained from G by identifying Z_n to a single vertex z_n in the vertex set of G_n^W. So G_n^W is a finite network.

Definition

Given $a \in S$, the escape probability from a is defined by

$$\lim_{n\to\infty} \mathbb{P}_a(\tau_{Z_n} < \tau_a^+).$$

Theorem

The escape probability is well-defined and is equal to $\mathbb{P}_a(\tau_a^+ = \infty)$.

Criterion for Recurrence

Definition

The effective conductance from a to ∞ is defined by

$$C_{eff}(a \leftrightarrow \infty) := C(a) \lim_{n\to\infty} \mathbb{P}_a(\tau_{Z_n} < \tau_a^+).$$

Corollary

A state $a \in S$ is recurrent if and only if $C_{eff}(a \leftrightarrow \infty) = 0$. Consequently, the corresponding reversible Markov chain is recurrent if and only if the effective conductance from any vertices to infinity is 0.

Rayleigh's Principle

We are almost ready to answer the question concerning random walk on $\mathbb{Z}^d$. The following lemma turns out to be extremely useful in practice.

Rayleigh's Monotonicity Principle

Let G be a connected graph (possibly infinite) with two weight function C and C' indicating the conductances on its edges with $C(e) \leq C'(e)$ for all $e \in E$.

1. If G is finite and A and Z are two nonempty disjoint subsets of S, then $C_{eff}(A \leftrightarrow Z) \leq C'_{eff}(A \leftrightarrow Z)$.
2. If G is infinite and a is one of its vertices, then $C_{eff}(a \leftrightarrow \infty) \leq C'_{eff}(a \leftrightarrow \infty)$.

Random Walk on Subgraphs

Since removing edges from a connected graph decrease the effective conductance, the following result is an immediate consequence of Rayleigh's Monotonicity Principle.

Corollary

Let the electrical network (G, C) correspond to an irreducible reversible Markov chain. If a simple random walk on G is recurrent, then so is the simple random walk on any connected subgraph of G.

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A simple random walk on a connected subgraph of $\mathbb{Z}^2$ is still recurrent. Our intuition is right!

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Proof of Pólya's Theorem

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- Adding and remove edges in $\mathbb{Z}^d$ so that the effective conductance $C_{eff}(0 \leftrightarrow \infty)$ becomes computable. See Chapter 19 in Klenke [2020] for more details.

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A: Yes!

- Adding and remove edges in $\mathbb{Z}^d$ so that the effective conductance $C_{\mathit{eff}}(0 \leftrightarrow \infty)$ becomes computable. See Chapter 19 in Klenke [2020] for more details.
- The Nash-Williams Criterion gives an easy condition on cutsets of a graph to verify recurrence. Energy and Random Path Method can be used to show transience. See Chapter 2 in Lyons and Peres [2016] and Lyons [1983] for more details.

Other Applications

Relating reversible Markov chains to electrical networks is a surprising but powerful technique in probability theory. Its applications have been found in the following stochastic processes and probabilistic models:

- Branching processes;
- Random walk on groups;
- Percolation;
- Random spanning trees;
- Discrete Gaussian free field;
- ...

Further References

Some further references are listed here for those of you who are interested in these topics:

1. For an introduction to random walks and electrical networks, see the short monograph by Doyle and Snell [1984].
2. For a comprehensive treatment to finite Markov chains and networks, see the book by Levin et al. [2017].
3. For an exhaustive discussion of reversible Markov chains and random walks on graphs, see the unpublished monograph by Aldous and Fill [2002].

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