HW 3.1: Prove directly from the definition that

$$I_t = \int_0^t s\, dB_s = tB_t - \int_0^t B_s\, ds \qquad (*)$$

Proof: Let $0 \le t_0 < t_1 \cdots < t_N = t$ be a partition of $[0,t]$. Let $\tau = \max_j |t_{j+1} - t_j|$.

By Def, we need $I_t = \lim_{\tau \downarrow 0} \sum_{j \ge 0} t_j \Delta B_j$.

Note that $\Delta(tB_t)_j := t_{j+1} B_{t_{j+1}} - t_j B_{t_j} = (t_{j+1} - t_j) B_{t_{j+1}} + t_j \Delta B_{t_j}$

Then $tB_t - 0B_0 = \sum_{j \ge 0} \Delta(tB_t)_j = \sum_j (\Delta t)_j B_{t_{j+1}} + \sum_j t_j \Delta B_{t_j}$. $(**)$

as a Riemann integral. $\downarrow N \uparrow \infty$

b.c. (B) is cts a.s. $\int_0^t B_s ds$ #.

HW 3.4 (ii) Check if $X_t = B_t^2$ is a mG.

ANS: No, it is NOT.

- X_t is $\mathcal{F}_t^B$-measurable;
- $\mathbb{E}|X_t| = \mathbb{E}B_t^2 = t < \infty$.

- $\mathbb{E}[X_t | \mathcal{F}_s^B] = \mathbb{E}[B_t^2 - B_s^2 + B_s^2 | \mathcal{F}_s^B] = \underbrace{\mathbb{E}[B_t^2 - B_s^2 | \mathcal{F}_s^B]}_{Z_t} + B_s^2 \neq B_s^2$

b.c. $\mathbb{E}[Z_t] = \mathbb{E}[B_t^2 - B_s^2] = t - s$.

HW 3.7 (a) Since the integrands are $f_n(t,\omega) = \frac{1}{n!} h_n\left(\frac{B_t}{\sqrt{t}}\right)$, they are $\begin{cases} \mathcal{B} \times \mathcal{F} \text{ measurable} \\ \mathcal{F}_t\text{-adapted.} \end{cases}$

To check the integrability, note that (wiki)

$$\mathbb{E}[h_n(N) h_m(N)] = \delta_{n,m} \sqrt{2\pi}\, n! \qquad \text{for } N \sim N(0,1).$$

Thus, $\mathbb{E}[f_n^2(t,\omega)] = \sqrt{2\pi}$ b.c. $B_t/\sqrt{t} \sim N(0,1)$.

Then $\mathbb{E}\left[\int_0^T f_n(t,\omega)^2 dt\right] = \sqrt{2\pi}\, T < \infty$

(b). $n=1$: $\int_0^t 1\, dB_u = B_t - 0 = t^{\frac{1}{2}} \frac{B_t}{\sqrt{t}}$ ✓.

$n=2$: $\int_0^t B_u\, dB_u = \frac{1}{2} B_t^2 - \frac{1}{2} t = \frac{1}{2!} t\, h_2\left(\frac{B_t}{\sqrt{t}}\right)$

$n=3$: $\int_0^t \left(\frac{1}{2} B_u^2 - \frac{1}{2} u\right) dB_u = \frac{1}{2} \int_0^t B_u^2\, dB_u - \frac{1}{2} \int_0^t u\, dB_u$

$$\overset{\text{exe 3.2}}{=} \frac{1}{6} B_t^3 - \cancel{\frac{1}{2}\int_0^t B_s ds} - \frac{1}{2} tB_t + \cancel{\frac{1}{2}\int_0^t B_s ds} = \frac{1}{3!} t^{\frac{3}{2}} h_3\left(\frac{B_t}{\sqrt{t}}\right)$$