AS.110.653 - Assignment 8

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Problem 8.2

Show that the solution $u(t,x)$ of the initial value problem

$$\frac{\partial u}{\partial t} = \frac{1}{2}\beta^2 x^2 \frac{\partial^2 u}{\partial x^2} + \alpha x \frac{\partial u}{\partial x} \qquad t > 0, x \in \mathbb{R}$$
$$u(0,x) = f(x), \qquad (f \in C_0^2(\mathbb{R}) \text{ given})$$

can be expressed as follows:

$$\begin{aligned} u(t,x) &= \mathbb{E}\left[f\left(x \cdot \exp\left\{\beta B_t + \left(\alpha - \frac{1}{2}\beta^2\right)t\right\}\right)\right] \\ &= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} f\left(x \cdot \exp\left\{\beta y + \left(\alpha - \frac{1}{2}\beta^2\right)t\right\}\right) \exp\left(-\frac{y^2}{2t}\right) \mathrm{d}y\,; \quad t > 0 \end{aligned}$$

Solution

We claim that this conclusion should follow directly from Kolmogorov's backward equation (Theorem 8.1.1, pp.141). We set up the appropriate structure:

We note that (using the 1-D Itô formula of Chapter 4)

$$\begin{aligned} X_t &= \beta B_t + \left(\alpha - \frac{1}{2}\beta^2\right)t \\ Y_t &= x \cdot \exp\left(\beta B_t + \left(\alpha - \frac{1}{2}\beta^2\right)t\right) \\ \mathrm{d}Y_t &= x \cdot \exp(X_t)\left(\alpha - \frac{1}{2}\beta^2\right)t + x \cdot \exp(X_t)\beta\,\mathrm{d}B_t + \frac{1}{2}x\exp(X_t)\beta^2\,\mathrm{d}t \\ &= Y_t\alpha\,\mathrm{d}t + Y_t\beta\,\mathrm{d}B_t \end{aligned} \tag{1}$$

The structure of Y_t thus satisfies the definition of an Itô diffusion (Def. 7.1.1, pp. 116), so Kolmogorov's theorem applies, and so

$$\begin{aligned} u(t,x) &= \mathbb{E}\left[f\left(x \cdot \exp\left\{\beta B_t + \left(\alpha - \frac{1}{2}\beta^2\right)t\right\}\right)\right] \\ &= \mathbb{E}^x[f(Y_t)] \end{aligned}$$

satisfies the equation

$$\frac{\partial u}{\partial t} = \mathcal{A}u$$

where $\mathcal{A}$ is the generator of Y_t. We can recover $\mathcal{A}$ this by (Theorem 7.3.3, pp 126) applied to equation (1):

$$\mathcal{A} = y\alpha\frac{\partial}{\partial y} + \frac{1}{2}\beta^2 y^2 \frac{\partial^2}{\partial y^2}$$

We have therefore recovered the original operator, and so we are done. Note that the uniqueness of $u(t,x)$ is also guaranteed by part (b) of Kolmogorov's Theorem, so there is no other $C^{1,1}(\mathbb{R} \times \mathbb{R})$ bounded solution to the IVP.

Problem 8.6

In connection with the deduction of the Black & Scholes formula for the price of an option (see Chapter 12) the following partial differential equation appears:

$$\begin{cases} \frac{\partial u}{\partial t} = -\rho u + \alpha x \frac{\partial u}{\partial x} + \frac{1}{2}\beta^2 x^2 \frac{\partial^2 u}{\partial x^2}; & t > 0, x \in \mathbb{R} \\ u(0,x) = (x-K)^+; & x \in \mathbb{R} \end{cases} \tag{2}$$

where $\rho > 0, \alpha, \beta$ and $K > 0$ are constants, and

$$(x-K)^+ = \max(x-K, 0)$$

Use the Feynman-Kac formula to prove that the solution u of this equation is given by

$$u(t,x) = \frac{e^{-\rho t}}{\sqrt{2\pi t}} \int_{\mathbb{R}} \left(x \cdot \exp\left\{ (\alpha - \frac{1}{2}\beta^2)t + \beta y \right\} - K \right)^+ e^{-\frac{y^2}{2t}} \, \mathrm{d}y \,; \qquad t > 0 \tag{3}$$

(This expression can be simplified further. See Exercise 12.13.)

Solution

The result will follow from the Feynman-Kac formula (Theorem 8.2.1, pp. 137). We may rewrite the solution $u(t,x)$ in terms of the quantities relevant to the theorem, and also utilize the result of Problem 8.2.

We see that:

$$\begin{aligned} u(t,x) &= \frac{e^{-\rho t}}{\sqrt{2\pi t}} \int_{\mathbb{R}} \left(x \cdot \exp\left\{ (\alpha - \frac{1}{2}\beta^2)t + \beta y \right\} - K \right)^+ e^{-\frac{y^2}{2t}} \, \mathrm{d}y \\ &= \mathbb{E}\left[\exp\left\{ -\int_0^t \rho \, \mathrm{d}s \right\} f(X_t) \right] \end{aligned}$$

where

$$\begin{aligned} X_t &= x \cdot \exp\left\{ (\alpha - \frac{1}{2}\beta^2)t + \beta B_t \right\} \\ f(x) &= (x-K)^+ \\ q(x) &= \rho \end{aligned}$$

From the previous problem we know that X_t is an Itô diffusion with generator $\mathcal{A}$:

$$\mathcal{A} = x\alpha \frac{\partial}{\partial x} + \frac{1}{2}\beta^2 x^2 \frac{\partial^2}{\partial x^2}$$

Furthermore $q(x)$ is continuous and bounded below, but $f(x) \notin C_0^2(\mathbb{R})$ as required by the specific version of the theorem. If we assume that we can extend to $f(x)$ being of at most polynomial growth[1], (which would still allow us to approximate it as a limit of smooth functions on which dominated convergence applies) we have that u will exactly satisfy

$$\begin{aligned} \frac{\partial u}{\partial t} &= \alpha u - \rho u \\ u(0,x) &= (x-K)^+ \end{aligned}$$

as desired.

[1] see for example http://www.stat.uchicago.edu/∼ lalley/Courses/391/Lecture12.pdf