110.653: Introduction to SDE

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1 Problem 7.2(c)

To tackle this problem, we need to infer from the expression for the infinitesimal generator associated with the Markov semigroup induced by Itô diffusion given in Theorem 7.3.3 of [1]. Given an Itô diffusion

$$\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}B_t.$$

If $f \in C_0^2(\mathbb{R}^n)$, then $f \in \mathcal{D}_A$ and

$$Af(x) = \sum_{i=1}^{n} b_i(x)\frac{\partial f}{\partial x_i} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(\sigma\sigma^\top)_{i,j}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}. \tag{1}$$

Now we want to find the corresponding drift and diffusion coefficients corresponding to the infinitesimal generator specified by

$$Af(x_1,x_2) = 2x_2\frac{\partial f}{\partial x_1} + \ln(1+x_1^2+x_2^2)\frac{\partial f}{\partial x_2} + \frac{1}{2}(1+x_1^2)\frac{\partial^2 f}{\partial x_1^2} + x_1\frac{\partial^2 f}{\partial x_1 \partial x_2} + \frac{1}{2}\frac{\partial^2 f}{\partial x_2^2}. \tag{2}$$

Let's first determine the column dimension m of the diffusion coefficient $\sigma \in \mathbb{R}^{2\times m}$. Suppose $m = 1$, we have

$$\frac{1}{2}\begin{bmatrix} \sigma_1^2(x_1,x_2) & \sigma_1(x_1,x_2)\sigma_2(x_1,x_2) \\ \sigma_1(x_1,x_2)\sigma_2(x_1,x_2) & \sigma_2^2(x_1,x_2) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(1+x_1^2) & \frac{1}{2}x_1 \\ \frac{1}{2}x_1 & \frac{1}{2} \end{bmatrix}.$$

It is equivalent to

$$\begin{bmatrix} \sigma_1^2(x_1,x_2) & \sigma_1(x_1,x_2)\sigma_2(x_1,x_2) \\ \sigma_1(x_1,x_2)\sigma_2(x_1,x_2) & \sigma_2^2(x_1,x_2) \end{bmatrix} = \begin{bmatrix} 1+x_1^2 & x_1 \\ x_1 & 1 \end{bmatrix}.$$

Equating the entries in the matrix equation above, it is easy to see it has no solutions. If $m = 2$, applying the same idea, we find that

$$\sigma\sigma^\top = \begin{bmatrix} \sigma_{11}^2+\sigma_{12}^2 & \sigma_{11}\sigma_{21}+\sigma_{12}\sigma_{22} \\ \sigma_{11}\sigma_{21}+\sigma_{12}\sigma_{22} & \sigma_{21}^2+\sigma_{22}^2 \end{bmatrix} = \begin{bmatrix} 1+x_1^2 & x_1 \\ x_1 & 1 \end{bmatrix}.$$

Certainly, the solution to the equation above is not unique. Let's just exhibit an obvious one by setting

$$\sigma_{11} = x_1, \sigma_{12} = 1, \sigma_{21} \equiv 1, \sigma_{22} \equiv 0,$$

that is

$$\sigma(x_1,x_2) = \begin{bmatrix} x_1 & 1 \\ 1 & 0 \end{bmatrix}.$$

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Comparing the first-order coefficients in (1) and (2), we have

$$b(x_1, x_2) = \begin{bmatrix} 2x_2 \\ \ln(1 + x_1^2 + x_2^2) \end{bmatrix}.$$

Thus the desired Itô process that we find is

$$\mathrm{d}X_t = \mathrm{d}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 2X_2 \\ \ln(1 + X_1^2 + X_2^2) \end{bmatrix} \mathrm{d}t + \begin{bmatrix} X_1 & 1 \\ 1 & 0 \end{bmatrix} \mathrm{d}B_t.$$

Since the problem only asks us to find an Itô process corresponding to the given infinitesimal generator, we are done.

2 Problem 7.4

(a). Given $x > 0$, consider the hitting time

$$\tau := \inf\{t > 0 : B_t = 0\} \text{ with } \mathbb{P}_x(B_0 = x) = 1.$$

For any $t \geq 0$, if $B_t < 0$, by the continuity of sample path, then $\tau \leq t$. By the reflection principle ,

$$\begin{aligned} \mathbb{P}_x(\tau \leq t) &= \mathbb{P}_x(\tau \leq t, B_t \geq 0) + \mathbb{P}_x(\tau \leq t, B_t < 0) \\ &= \mathbb{P}_x(\tau \leq t, B_t \geq 0) + \mathbb{P}_x(B_t < 0) \\ &= \frac{1}{2}\mathbb{P}_x(\tau \leq t) + \mathbb{P}_x(B_t < 0). \end{aligned}$$

From the derivation above, the cumulative distribution function of τ under $\mathbb{P}_x$ is

$$\begin{aligned} \mathbb{P}_x(\tau \leq t) &= 2\mathbb{P}_x(B_t < 0) \\ &= 2\frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{-x} e^{-y^2/2t} \, \mathrm{d}y \\ &= \sqrt{\frac{2}{\pi t}} \int_{x}^{\infty} e^{-y^2/2t} \, \mathrm{d}y \\ &= \sqrt{\frac{2}{\pi}} \int_{x/\sqrt{t}}^{\infty} e^{-z^2/2} \, \mathrm{d}z \end{aligned}$$

By dominated convergence theorem and continuity of probability measure, one has

$$\mathbb{P}_x(\tau < \infty) = \lim_{t \to \infty} \mathbb{P}_x(\tau \leq t) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} e^{z^2/2} \, \mathrm{d}z.$$

Evaluating the last Gaussian integral, we can conclude that $\mathbb{P}_x(\tau < \infty) = 1$.

(b). Exploiting the martingale $(M_t)_{t \geq 0}$ defined by

$$M_t := \exp(u(B_t - x) - u^2 t/2) \quad \text{for } u \in \mathbb{R}.$$

Indeed, for $0 \leq s \leq t < \infty$

$$\begin{aligned} \mathbb{E}_x[M_t \,|\, \mathscr{F}] &= \mathbb{E}_x[\exp(u(B_t - B_s)) \,|\, \mathscr{F}_s] e^{u(B_s - x) - u^2 t/2} \\ &= \exp\left(\frac{1}{2}u^2(t-s) + u(B_s - x) - \frac{1}{2}u^2 t\right) \\ &= M_s. \end{aligned}$$

Applying the optional sample theorem to the bounded stopping time $\tau \wedge t$, we have $\mathbb{E}_x[M_{\tau\wedge t}] = \mathbb{E}_x[M_0] = 1$. Hence

$$\mathbb{E}_x\left[\exp(uB_{\tau\wedge t} - ux - u^2(t \wedge \tau)/2)\right] = 1. \quad , \forall t.$$

Assume $u < 0$, the integrand is bounded almost surely, by bounded convergence theorem,

$$\mathbb{E}_x[\exp(-ux - u^2\tau/2)] = 1 \quad \text{implies} \quad \mathbb{E}_x[\exp(-u^2\tau/2)] = e^{ux}.$$

Let $u = -\sqrt{2s}$ for some $s > 0$, then the Laplace transform of τ is given by

$$\mathbb{E}_x[e^{-s\tau}] = e^{-\sqrt{2s}x}.$$

It is well-known that differentiating under the integral sign is legitimate for Laplace transform. So

$$\mathbb{E}_x[\tau e^{-s\tau}] = e^{-\sqrt{2s}x}\sqrt{2}x\frac{1}{2}\frac{1}{\sqrt{s}}.$$

It follows from monotone convergence theorem that $\mathbb{E}_x[\tau] = \lim_{s\to 0} e^{-\sqrt{2s}x}\sqrt{2}x\frac{1}{2}\frac{1}{\sqrt{s}} = \infty$.

3 Problem 7.18

(a). *Proof.* Since $f \in \mathcal{D}_A \subseteq C^2$, then so is $g(x) := (f(x) - f(a))/(f(b) - f(a))$. Applying Dynkin's formula to the bounded stopping time $\tau \wedge t$ and g, we have

$$\mathbb{E}_x[g(X_{\tau\wedge t})] = g(x) + \mathbb{E}_x\left[\int_0^{\tau\wedge t} \mathcal{A}f(X_s)\,\mathrm{d}s\right] = g(x),$$

where the integral term vanishes due to the fact that $\mathcal{A}f = 0$. Since g is a bounded between 0 and 1 on $[a, b]$, then, by bounded convergence theorem and the fact that $\tau < \infty$ almost surely,

$$\mathbb{E}_x[g(X_\tau)] = \frac{f(x) - f(a)}{f(b) - f(a)}.$$

As $g(a) = 0$ and $g(b) = 1$ and the sample path is continuous, we derive that

$$\mathbb{E}_x[g(X_\tau)] = \mathbb{P}_x(X_\tau = a) \cdot 0 + \mathbb{P}_x(X_\tau = b) \cdot 1 = \mathbb{P}_x(X_\tau = b).$$

Combining the equations above, we can conclude that

$$p := \mathbb{P}_x(X_\tau = b) = \frac{f(x) - f(a)}{f(b) - f(a)}.$$ □

(b). *Proof.* If $X_t = x + B_t$, it satisfies the SDE $\mathrm{d}X_t = \mathrm{d}B_t$, then

$$\mathcal{A}f(x) = \frac{1}{2}f''(x).$$

Choose $f(x) = x$, it is easy to see that $\mathcal{A}f(x) = 0$ for this f in particular. Applying part (a), we conclude that

$$p = \frac{x - a}{b - a}.$$ □

(c). *Proof.* If $X_t = x + \mu t + \sigma B_t$, then, for $f \in C^2$,

$$\mathcal{A}f(x) = \mu f'(x) + \frac{1}{2}\sigma^2 f''(x).$$

Put $\mathcal{A}f = 0$, we find the solution to this ODE is

$$C_1e^{0\cdot x} + C_2e^{-\frac{2\mu}{\sigma^2}x} = C_1 + C_2e^{-\frac{2\mu}{\sigma^2}x}.$$

For simplicity, choose $C_1 = 0$ and $C_2 = 1$. Substituting $f(x) = e^{-\frac{2\mu}{\sigma^2}x}$ into part (a), we can conclude that

$$p = \frac{e^{-\frac{2\mu}{\sigma^2}x} - e^{-\frac{2\mu}{\sigma^2}a}}{e^{-\frac{2\mu}{\sigma^2}b} - e^{-\frac{2\mu}{\sigma^2}a}}.$$ □

References

[1] Øksendal, B. (2013). *Stochastic Differential Equations*, 6th ed. Springer-Verlag, Berlin Heidelberg.