AS.110.653 - Assignment 5

George A. Kevrekidis

October 6, 2021

Problem 1

Consider the SDE with smooth bounded a, b:

$$\mathrm{d}X_t = a(X_t)\,\mathrm{d}t + b(X_t)\,\mathrm{d}W_t \tag{1}$$

Derive the strong order 1.5 Itô-Taylor scheme (ref. the class notes or Kloeden+Platen Chapter 10). [Hint: you will need to use the multiple integrals, paying attention to their relations:]

$$\begin{aligned}
I_1 &= \int_0^h \mathrm{d}W_s \\
I_{01} &= \int_0^h \int_0^s \mathrm{d}W_{s_1}\,\mathrm{d}s \\
I_{11} &= \int_0^h \int_0^s \mathrm{d}W_{s_1}\,\mathrm{d}W_s \\
I_{111} &= \int_0^h \int_0^s \int_0^r \mathrm{d}W_u\,\mathrm{d}W_r\,\mathrm{d}W_s
\end{aligned}$$

Solution

We follow the process described in the notes:

After applying the Itô-Taylor expansion to the integral form of Equation (1) we have:

$$\begin{aligned}
X_t = X_0 + a(X_0)t + b(X_0)\underbrace{(W_t - W_0)}_{I_1} &+ [b'b]\,(X_0)\underbrace{\int_0^t \int_0^s \mathrm{d}W_r\,\mathrm{d}W_s}_{I_{11}} \\
+ [b'a + \frac{1}{2}b''b^2](X_0)\underbrace{\int_0^t s\,\mathrm{d}W_s}_{I_{01}} &+ [a'b]\,(X_0)\underbrace{\int_0^t (W_s - W_0)\,\mathrm{d}s}_{I_{10}} \\
+ [(b'b)'b]\,(X_0)\underbrace{\int_0^t \int_0^s \int_0^r \mathrm{d}W_u\,\mathrm{d}W_r\,\mathrm{d}W_s}_{I_{111}} &+ \left[a'a + \frac{1}{2}a''b\right](X_0)\frac{1}{2}t^2 + o(t^{3/2}))
\end{aligned}$$

Ignoring the last term (which tends to 0) leaves us with the more compact equation

$$\begin{aligned}
X_t = X_0 &+ a(X_0)t + b(X_0)I_1 + [b'b]\,(X_0)I_{11} \\
&+ [b'a + \frac{1}{2}b''b^2](X_0)I_{01} + [a'b]\,(X_0)I_{10} \\
&+ [(b'b)'b]\,(X_0)I_{111} + \left[a'a + \frac{1}{2}a''b\right](X_0)\frac{1}{2}t^2
\end{aligned} \tag{2}$$

The first four terms are familiar since they are shared by the EM and Milstein schemes, and give us

$$Y_{n+1} = \underbrace{Y_n + a(Y_n)\Delta t + b(Y_n)\Delta W_n}_{\text{Euler Maruyama}} + \underbrace{[b'b(X_{t_n})\frac{1}{2}(\Delta W_n^2 - \Delta t)]}_{\text{Milstein}} + \dots \tag{3}$$

Now we simplify the remaining four terms. It is clear that for the last term

$$\boxed{\left[a'a + \frac{1}{2}a''b\right](X_0)\frac{1}{2}t^2 \mapsto \left[a'a + \frac{1}{2}a''b\right](Y_n)\frac{1}{2}\Delta t^2} \tag{4}$$

The remaining terms have the stochastic integrals. Recall (ref. Example 4.1.4, pp. 45) that

$$\int_0^t s\,\mathrm{d}B_s = tB_t - \int_0^t B_s\,\mathrm{d}s$$

Thus, we have

$$\begin{aligned} I_{01} &= \int_0^t s\,\mathrm{d}W_s \\ &= tW_t - \int_0^t W_s\,\mathrm{d}s \\ &= tW_t - \Delta Z \end{aligned}$$

where, for convenience, we will define

$$\Delta Z = \int_0^t W_s\,\mathrm{d}s$$

which then translates to:

$$\boxed{\left[b'a + \frac{1}{2}b''b^2\right](X_0)I_{01} \mapsto \left[b'a + \frac{1}{2}b''b^2\right](Y_n)(\Delta t\Delta W_t - \Delta Z)} \tag{5}$$

Continuing,

$$\begin{aligned} I_{10} &= \int_0^t (W_s - W_0)\,\mathrm{d}s \\ &= \int_0^t (W_s)\,\mathrm{d}s \\ &= \Delta Z \end{aligned}$$

and so

$$\boxed{[a'b](X_0)I_{10} \mapsto [a'b]\,(Y_n)\Delta Z} \tag{6}$$

Finally:

$$\begin{aligned} I_{111} &= \int_0^t \int_0^s \int_0^r \mathrm{d}W_u\,\mathrm{d}W_r\,\mathrm{d}W_s \\ &= \int_0^t I_{11}\,\mathrm{d}W_s \\ &= \int_0^t \frac{1}{2}\big((W_s - W_0)^2 - s\big)\,\mathrm{d}W_s \\ &= \frac{1}{2}\int_0^t W_s^2\,\mathrm{d}W_s - \frac{1}{2}\int_0^t s\,\mathrm{d}W_s \\ &= \frac{1}{2}\left(\frac{1}{3}W_t^3 - \int_0^t W_s\,\mathrm{d}s\right) - \frac{1}{2}tW_t + \frac{1}{2}\int_0^t W_s\,\mathrm{d}s \\ &= \frac{1}{6}W_t^3 - \frac{1}{2}tW_t \end{aligned}$$

where we have used (see Problem 4.2, pp. 55) that using the 1-d Itô formula:

$$\mathrm{d}\left(\frac{1}{3}B_t^3\right) = B_t\,\mathrm{d}t + B_t^2\,\mathrm{d}B_t$$

$$\Longleftrightarrow$$

$$\frac{1}{3}B_t^3 = \int_0^t B_s^2\,\mathrm{d}B_s + \int_0^t B_s\,\mathrm{d}s$$

So we have:

$$\boxed{[(b'b)'b]\,(X_0)I_{111} \mapsto [(b'b)'b]\,(Y_n)\left(\frac{1}{6}\Delta W_t^3 - \frac{1}{2}\Delta t\Delta W_t\right)} \tag{7}$$

combining the results (equations 3-7) allows us to write the strong 1.5 order scheme of equation (2) as

$$\begin{aligned} Y_{n+1} &= Y_n + a(Y_n)\Delta t + b(Y_n)\Delta W_n + [b'b(X_{t_n})\frac{1}{2}(\Delta W_n^2 - \Delta t)] \\ &+ \left[b'a + \frac{1}{2}b''b^2\right](Y_n)(\Delta t\Delta W_t - \Delta Z) + [a'b]\,(Y_n)\Delta Z \\ &+ [(b'b)'b]\,(Y_n)\left(\frac{1}{6}\Delta W_t^3 - \frac{1}{2}\Delta t\Delta W_t\right) + \left[a'a + \frac{1}{2}a''b\right](Y_n)\frac{1}{2}\Delta t^2 \end{aligned}$$

where

$$\Delta W_n = W_{t_{n+1}-W_{t_n}} \sim \sqrt{\Delta t}\xi_n$$

with $\xi_n \sim \mathcal{N}(0,1)$.

We are still left with ΔZ (which we in principle need to simulate if we are to apply the numerical scheme): Since the quantities in

$$I_{10} = tI_1 - I_{01}$$

(ex. 4.1.4) are all gaussian, we can describe ΔZ in terms of it's mean, variance, and correlation to I_1 (which we have already sampled).

$$\begin{aligned} \mathbb{E}[\Delta Z] &= \mathbb{E}\left[\int_0^t W_s\,\mathrm{d}s\right] \\ &= 0 \end{aligned}$$

Then

$$\begin{aligned} \mathbb{E}[I_{10}I_1] &= \mathbb{E}\left[\int_0^t W_s\,\mathrm{d}s\,W_t\right] \\ &= \int_0^t \mathbb{E}[W_sW_t]\,\mathrm{d}s \\ &= \int_0^t s\,\mathrm{d}s \\ &= \frac{1}{2}t^2 \\ \mathbb{E}\left[I_{10}^2\right] &= \mathbb{E}\left[\left(\int_0^t W_s\,\mathrm{d}s\right)^2\right] \\ &= \int_0^t s^2\,\mathrm{d}s \\ &= \frac{t^3}{3} \end{aligned}$$

where for the first result we have used Problem 2.8, pp. 17, for the second result we have used that $\mathbb{E}[W_s W_t] = s(= s \wedge t)$ and the third result follows by the Itô Isometry.

Thus the variance $\mathbb{V}[I_{10}^2] = \frac{t^3}{3}$. Since we want to keep the correlation between I_1 and I_{10} when generating our samples, we will write:

$$\Delta Z_n \sim \frac{1}{2}(\Delta t)^{3/2}\left(\xi_n + \frac{1}{\sqrt{3}}\zeta_n\right)$$

where $\zeta_n \sim \mathcal{N}(0,1)$ as well. One can check that the variance of the above expression is indeed:

$$\frac{(\Delta t^3)}{4}\left(\frac{4}{3}\right) = \frac{(\Delta t)^3}{3}$$

where by independence of ξ_n, ζ_n the variances are additive, while retaining the desired correlation:

$$\mathbb{E}[\Delta_Z I_1] = \mathbb{E}\left[\frac{1}{2}(\Delta t)^{3/2}(\Delta t)^{1/2}\xi_n^2\right] = \frac{1}{2}(\Delta t)^2 = \mathbb{E}[I_{10} I_1]$$

$\square$

Problem 2

Consider the Ornstein-Uhlenbeck equation with $\lambda < 0$*:*

$$\mathrm{d}X_y = \lambda X_t \,\mathrm{d}t + \sigma \,\mathrm{d}W_t \tag{8}$$

1. *Find the range of the time step size* δ *such that the Euler-Maruyama scheme*

$$Y_{n+1} = Y_n + \lambda Y_n \delta + \sigma\sqrt{\delta}\xi_n; \qquad Y_0 = 0$$

   where $\xi_n \sim \mathcal{N}(0,1)$*, is stable in the sense that* $\mathbb{E}[Y_n^2] < \infty$ *for all* n *and compute the* $\lim_{n\to\infty} \mathbb{E}[Y_n^2]$.

2. *Find the range of the time step size* δ *so that the implicit Euler scheme*

$$Y_{n+1} = Y_n + \lambda Y_{n+1} \delta + \sigma\sqrt{\delta}\xi_n; \qquad Y_0 = 0$$

   where $\xi_n \sim \mathcal{N}(0,1)$*, is stable in the sense that* $\mathbb{E}[Y_n^2] < \infty$ *for all* n *and compute the* $\lim_{n\to\infty} \mathbb{E}[Y_n^2]$

Solution

We rewrite the EM scheme as

$$\begin{aligned}
Y_{n+1} &= (1+\lambda\delta)Y_n + \sigma\sqrt{\delta}\xi_n \\
&= (1+\lambda\delta)\Big((1+\lambda\delta)Y_{n-1} + \sigma\sqrt{\delta}\xi_{n-1}\Big) + \sigma\sqrt{\delta}\xi_n \\
&= (1+\lambda\delta)^2 Y_{n-1} + \sigma\sqrt{\delta}(\xi_n + (1+\lambda\delta)\xi_{n-1}) \\
&= \ldots \\
&= (1+\lambda\delta)^{n+1} Y_0 + \sigma\sqrt{\delta}\sum_{j=0}^{n}(1+\lambda\delta)^j \xi_{n-j}
\end{aligned}$$

Note that $\mathbb{E}[Y_n]$ is finite for all n since by linearity of expectation and the zero initial condition this will always be trivial. So we consider the second moment:

$$\mathbb{E}\big[Y_{n+1}^2\big] = \mathbb{E}\left[\left((1+\lambda\delta)^{n+1} Y_0 + \sigma\sqrt{\delta}\sum_{j=0}^{n}(1+\lambda\delta)^j \xi_{n-j}\right)^2\right]$$

This simplifies dramatically since $Y_0 = 0$ and from the independence of the ξ_j. We are left with:

$$\begin{aligned}
\mathbb{E}\big[Y_{n+1}^2\big] &= \mathbb{E}\left[\left(\sigma\sqrt{\delta}\sum_{j=0}^{n}(1+\lambda\delta)^j \xi_{n-j}\right)^2\right] \\
&= \sigma^2\delta\mathbb{E}\left[\sum_{j=0}^{n}(1+\lambda\delta)^{2j}\xi_{n-j}^2\right] \\
&= \sigma^2\delta\sum_{j=0}^{n}(1+\lambda\delta)^{2j}\mathbb{E}[\xi_{n-j}^2]
\end{aligned}$$

Then, since the variance of the ξ_i is known, we are left with a geometric series in $(1+\lambda\delta)^2$ which converges if and only if

$$\left|(1+\lambda\delta)^2\right| < 1$$

or

$$-1 < (1 + \lambda\delta) < 1$$

which corresponds to

$$0 < \delta < -\frac{2}{\lambda}$$

Now, if for an appropriate choice of δ this limit exists, we will have:

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{E}\left[Y_{n+1}^2\right] &= \lim_{n\to\infty} \left[\sigma^2 \delta \sum_{j=0}^{n} (1 + \lambda\delta)^{2j}\right] \\
&= \frac{\sigma^2 \delta}{1 - (1 + \lambda\delta)^2}
\end{aligned}$$

□

Now consider *Case 2*. Grouping the Y_{n+1} terms of the implicit scheme together, and then solving for them gives us the equivalent equation

$$\begin{aligned}
Y_{n+1} &= (1 - \lambda\delta)^{-1} \left[Y_n + \sigma\sqrt{\delta}\xi_n\right] \\
&= (1 - \lambda\delta)^{-1} \left[(1 - \lambda\delta)^{-1} \left(Y_{n-1} + \sigma\sqrt{\delta}\xi_{n-1}\right) + \sigma\sqrt{\delta}\xi_n\right] \\
&= \ldots \\
&= (1 - \lambda\delta)^{-(n+1)} Y_0 + \sigma\sqrt{\delta} \sum_{j=0}^{n} (1 - \lambda\delta)^{-j-1} \xi_{n-j} \\
&= \sigma\sqrt{\delta} \sum_{j=0}^{n} (1 - \lambda\delta)^{-j-1} \xi_{n-j}
\end{aligned}$$

Since $Y_0 = 0$. Again, $\mathbb{E}[Y_n] = 0$ by linearity of expectation and the independence of the ξ_i. Then,

$$\mathbb{E}\left[Y_{n+1}^2\right] = \sigma^2 \delta \sum_{j=0}^{n} (1 - \lambda\delta)^{-2(j+1)}$$

again using the same argument as in *Case 1*, which is a geometric series in $(1 - \lambda\delta)^{-2}$. This time however, because $\delta > 0, \lambda < 0$ the quantity $(1 - \lambda\delta) > 1$ all (positive) δ, and hence the geometric series will converge for any value of the step size. Using the closed form of the geometric series limit gives us:

$$\lim_{n\to\infty} \mathbb{E}\left[Y_{n+1}^2\right] = \frac{(1 - \lambda\delta)^{-2} \sigma^2 \delta}{1 - (1 - \lambda\delta)^{-2}}$$

One may alternatively show this using the recursion scheme directly, without the geometric series results.

□