Stochastic ODEs HW3

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Problem 4.4

(a) Let X_t be the Ito process defined via

$$dX_t := \theta(t,\omega)dB(t) - \frac{1}{2}\theta^2(t,\omega)dt.$$

Then note that $Z_t = e^{X_t}$ by definition. Let $g(t,x) = e^x$. Then by Theorem 4.1.2., Z_t is again an Ito process and

$$\begin{aligned}
dZ_t &= \frac{\partial g}{\partial t}(t,X_t)dt + \frac{\partial g}{\partial x}(t,X_t)dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t,X_t)\cdot(dX_t)^2 \\
&= 0 + e^{X_t}dX_t + \frac{1}{2}e^{X_t}(dX_t)^2 \\
&= Z_t\bigg(\theta(t,\omega)dB(t) - \frac{1}{2}\theta^2(t,\omega)dt\bigg) + \frac{1}{2}Z_t\bigg(\theta(t,\omega)dB(t) - \frac{1}{2}\theta^2(t,\omega)dt\bigg)^2 \\
&= Z_t\bigg(\theta(t,\omega)dB(t) - \frac{1}{2}\theta^2(t,\omega)dt\bigg) + \frac{1}{2}Z_t\theta^2(t,\omega)dt \\
&= Z_t\theta(t,\omega)dB(t),
\end{aligned}$$

where I have used the rules $dB_t \cdot dB_t = dt$ and $dt \cdot dt = dt \cdot dB_t = dB_t \cdot d_t = 0$, and the final line is because the two terms match.

(b) By the formula in part (a), it holds that

$$\begin{aligned}
Z_t &= \int_0^t Z_s\theta(s,\omega)dB(s) \\
&= \int_0^t \sum_{k=1}^n Z_s\theta_k(s,\omega)dB_k(s) \\
&= \sum_{k=1}^n \int_0^t Z_s\theta_k(s,\omega)dB_k(s).
\end{aligned}$$

It is straightforward to check that a sum of martingales is a martingale, and hence by Corollary 3.2.6 Z_t is a martingale by the assumption that $Z_t\theta_k(t,\omega) \in \mathcal{V}[0,T]$.

Problem 4.6

(a) Let Z_t be the process defined via

$$dZ_t = cdt + \alpha dB_t.$$

Then $X_t = e^{Z_t}$, so by the Ito Formula (with $g(t,z) = e^z$ just as in the previous problem),

$$\begin{aligned}
dX_t &= \frac{\partial g}{\partial t}(t, Z_t)dt + \frac{\partial g}{\partial z}(t, Z_t)dZ_t + \frac{1}{2}\frac{\partial^2 g}{\partial z^2}(t, Z_t) \cdot (dZ_t)^2 \\
&= 0 + e^{Z_t}dZ_t + \frac{1}{2}e^{Z_t}(dZ_t)^2 \\
&= X_t(cdt + \alpha dB_t) + \frac{1}{2}X_t(cdt + \alpha dB_t)^2 \\
&= X_t(cd_t + \alpha dB_t) + \frac{1}{2}X_t\alpha^2 dt \\
&= X_t(c + \frac{1}{2}\alpha^2)dt + \alpha X_t dB_t.
\end{aligned}$$

(b) Let $g(t,x) = \exp(ct + \alpha^\top x)$ for $x \in \mathbb{R}^n$, and $\alpha \in \mathbb{R}^n$. We collect derivatives:

$$\begin{aligned}
\frac{\partial g}{\partial t} &= c\exp(ct + \alpha^\top x); \\
\frac{\partial g}{\partial x_i} &= \alpha_i \exp(ct + \alpha^\top x); \\
\frac{\partial^2 g}{\partial x_i \partial x_j} &= \alpha_i\alpha_j \exp(ct + \alpha^\top x).
\end{aligned}$$

By Theorem 4.2.1 (the multidimensional Ito formula),

$$\begin{aligned}
dX_t &= \frac{\partial g}{\partial t}(t, B)dt + \sum_i \frac{\partial g}{\partial x_i}(t, B)dB_i + \frac{1}{2}\sum_{i,j}\frac{\partial^2 g}{\partial x_i \partial x_j}(t, B)dB_i dB_j \\
&= c\exp(ct + \alpha^\top B)dt + \sum_i \alpha_i \exp(ct + \alpha^\top B)dB_i + \frac{1}{2}\sum_{i,j}\alpha_i\alpha_j \exp(ct + \alpha^\top B)dB_i dB_j \\
&= cX_t dt + \sum_i \alpha_i X_t dB_j + \frac{1}{2}\sum_i \alpha_i^2 X_t dB_i^2 + \frac{1}{2}\sum_{i \neq j}\alpha_i\alpha_j X_t dB_i dB_j \\
&= cX_t dt + \sum_i \alpha_i X_t dB_j + \frac{1}{2}\sum_i \alpha_i^2 X_t dt \\
&= \left(c + \frac{1}{2}\sum_j \alpha_j^2\right)X_t dt + X_t\left(\sum_j \alpha_j dB_j\right),
\end{aligned}$$

where I have used that $dB_j dB_i = \delta_{ij}dt$ as well as the standard rules.

Problem 4.15

Set $g(t,z) := (x^{1/3} + \frac{1}{3}z)^3$. By the Ito Formula,

$$\begin{aligned}
dX_t &= \frac{\partial g}{\partial t}(t, B_t)dt + \frac{\partial g}{\partial x}(t, B_t)dB_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, B_t) \cdot (dB_t)^2 \\
&= 0 + (x^{1/3} + \frac{1}{3}B_t)^2 dB_t + \frac{1}{2}\frac{2}{3}(x^{1/3} + \frac{1}{3}B_t)(dB_t)^2 \\
&= (x^{1/3} + \frac{1}{3}B_t)^2 dB_t + \frac{1}{2}\frac{2}{3}(x^{1/3} + \frac{1}{3}B_t)dt \\
&= X_t^{2/3}dB_t + \frac{1}{3}X_t^{1/3}dt
\end{aligned}$$

and $X_0 = x$ since $B_0 = 0$.