110.653: Introduction to SDE: HW1 solution

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1 Problem 2.8

Proof.

1. According to (2.2.3) in [2], the characteristic function under the probability measure $\mathbb{P}_x$ (an n-dimensional Gaussian process starts at x) of the random vector
$$\mathbf{Z} = (B_{t_1}, \ldots, B_{t_k})^\top \in \mathbb{R}^{nk}$$
is
$$\mathbb{E}_x\big[\exp\big(i\mathbf{u}^\top \mathbf{Z}\big)\big] = \exp\bigg(-\frac{1}{2}\mathbf{u}^\top \mathbf{C}\mathbf{u} + i\mathbf{u}^\top \mathbf{M}\bigg), \tag{1}$$
where $\mathbf{u} := (u_1, \ldots, u_{nk}) \in \mathbb{R}^{nk}$, the matrix $\mathbf{C} \in \mathbb{R}^{nk\times nk}$ is the covariance matrix of $\mathbf{Z}$ and $\mathbf{M} = \mathbb{E}_x[\mathbf{Z}]$.
Now choose $k = 1$, $n = 1$, $t_1 = t$ and $x = 0$ in (1). In this case $\mathbf{M} = \mathbb{E}_0[\mathbf{Z}] = \mathbb{E}_0[B_t] = 0$. The covariance matrix $\mathbf{C}$ is also a scalar which is exactly the variance of B_t, that is $\mathrm{Var}(B_t) = t$. Hence
$$\mathbb{E}_0[\exp(iuB_t)] = \exp\bigg(-\frac{1}{2}u^2 t + 0\bigg) = \exp\bigg(-\frac{1}{2}u^2 t\bigg) \quad \text{for all } u \in \mathbb{R}. \tag{2}$$

2. As the author doesn't ask for a rigorous proof for this part, I will follow the steps suggested by the author without careful justification, but a rigorous proof can be done by invoking for example Theorem 6.4.1 in [1]. Consider the Taylor expansion on both sides of (2), one finds that
$$\sum_{m=0}^{\infty} \frac{(iu)^m}{m!}\mathbb{E}_0[B_t^m] = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k k!}u^{2k}t^k.$$
By comparing the coeffients of u^m when m is an even number, we have
$$(-1)^k \frac{1}{(2k)!}\mathbb{E}_0[B_t^{2k}] = \frac{(-1)^k}{2^k k!}t^k \qquad \Longrightarrow \qquad \mathbb{E}_0[B_t^{2k}] = \frac{(2k)!}{2^k k!}t^k \tag{3}$$
for all $k \in \mathbb{N}$.

3. Since the marginal distribution of B_t is $\mathcal{N}(0, t)$, it is immediate that
$$\mathbb{E}_0[f(B_t)] = \frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} f(x)e^{-\frac{x^2}{2t}}\,\mathrm{d}x \tag{4}$$
for all Borel function f. Let $f(x) = x^{2k}$, we prove the desired result by induction on $k \in \mathbb{N}$.
   - **Base case:** When $k = 1$, then
$$\mathbb{E}_0[B_t^2] = \mathrm{Var}(B_t) + (\mathbb{E}_0[B_t])^2 = t = \frac{2!}{2}t.$$

- **Induction hypothesis:** Suppose the formula holds for $k \leqslant m$ for $m \in \mathbb{N}$.
- **Induction step:** For $k = m+1$, using (4) and integration by parts, we have

$$\begin{aligned}
\mathbb{E}_0\Big[B_t^{2(m+1)}\Big] &= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} x^{2m+2} e^{-\frac{x^2}{2t}} \,\mathrm{d}x \\
&= \frac{-t}{\sqrt{2\pi t}} (-1) \int_{\mathbb{R}} e^{-\frac{x^2}{2m}} (2m+1) x^{2m} \,\mathrm{d}x \\
&= \frac{t(2m+1)}{\sqrt{2\pi t}} \int_{\mathbb{R}} x^{2m} e^{-\frac{x^2}{2m}} \,\mathrm{d}x \\
&= t(2m+1) \frac{(2m)!}{2^m m!} t^m \\
&= \frac{2(m+1)(2m+1)}{2(m+1)} \frac{(2m)!}{2^m m!} t^{m+1} \\
&= \frac{(2(m+1))!}{2^{m+1}(m+1)!} t^{m+1}.
\end{aligned}$$

  This completes the induction step.

Thus the desired result follows from the principles of mathematical induction.

4. In this part, the Brownian motion $(B_t)_{t \geqslant 0}$ takes value in $\mathbb{R}^n$ and starts at x. We shall show the formula
$$\mathbb{E}_x[|B_t - B_s|^4] = n(n+2)|t-s|^2 \quad \text{for all } t, s \in \mathbb{R}_{\geqslant 0}, \tag{5}$$
where $|\cdot|$ is the Euclidean distance in $\mathbb{R}^n$. This can also be done by induction on $n \in \mathbb{N}$. For the case $t = s$, the desired formula (5) is trivial, it suffices to prove the case for $0 \leqslant s < t < \infty$. The case for $0 \leqslant t < s < \infty$ follows from the preceding case and the symmetric roles that s and t play in the expression given in (5). Before we dive into the induction procedures, let's first simplify the left-hand side of (5) using Markov property. Let $(\mathcal{F}_t)_{t \geqslant 0}$ be the natural filtration generated by $(B_t)_{t \geqslant 0}$. Hence

$$\begin{aligned}
\mathbb{E}_x[|B_t - B_s|^4] &= \mathbb{E}_x[\mathbb{E}_x[|B_t - B_s|^4 \,|\, \mathcal{F}_s]] \\
&= \mathbb{E}_x[\mathbb{E}_{B_s}[|B_{t-s} - B_0|^4]] \\
&= \frac{1}{(2\pi s)^{n/2}} \int_{\mathbb{R}} \mathbb{E}_y[|B_{t-s} - y|^4] \, e^{-\frac{|y-x|^2}{2s}} \,\mathrm{d}y \\
&= \frac{1}{(2\pi s)^{n/2}} \int_{\mathbb{R}} \frac{1}{(2\pi(t-s))^{n/2}} \int_{\mathbb{R}} |z-y|^4 e^{-\frac{|z-y|^2}{2(t-s)}} \,\mathrm{d}z \, e^{-\frac{|y-x|^2}{2s}} \,\mathrm{d}y \\
&= \frac{1}{(2\pi s)^{n/2}} \int_{\mathbb{R}} e^{-\frac{|v|^2}{2s}} \,\mathrm{d}v \, \frac{1}{(2\pi(t-s))^{n/2}} \int_{\mathbb{R}} |u|^4 e^{-\frac{|u|^2}{2(t-s)}} \,\mathrm{d}u \\
&= \frac{1}{(2\pi(t-s))^{n/2}} \int_{\mathbb{R}} |u|^4 e^{-\frac{|u|^2}{2(t-s)}} \,\mathrm{d}u,
\end{aligned}$$

where the second equality from bottom follows from changes of variables and translation invariance of Lebesgue integration. The last display shows that $\mathbb{E}_x[|B_t - B_s|^4] = \mathbb{E}_0[|B_{t-s}|^4]$. Denote the kth coordinate of the Brownian motion at time t by $B_t^{(k)}$.

- **Base case:** When $n = 1$, setting $k = 2$ in (3) shows that
$$\mathbb{E}_0[|B_{t-s}|^4] = \frac{4!}{2^2 2!}(t-s)^2 = 3|t-s|^2 = 1 \cdot (1+2)|t-s|^2.$$
- **Induction hypothesis:** Suppose (5) holds for $n \leqslant m$ for $m \in \mathbb{N}$.
- **Induction step:** For $n = m + 1$, we have
$$\begin{aligned}
\mathbb{E}_0[|B_{t-s}|^4] &= \mathbb{E}_0\Bigg[\bigg(\sum_{k=1}^{m+1} \big(B_{t-s}^{(k)}\big)^2\bigg)^2\Bigg] \\
&= \mathbb{E}_0\Bigg[\bigg(\sum_{k=1}^{m} \big(B_{t-s}^{(k)}\big)^2 + \big(B_{t-s}^{(m+1)}\big)^2\bigg)^2\Bigg] \\
&= \mathbb{E}_0\Bigg[\bigg(\sum_{k=1}^{m} \big(B_{t-s}^{(k)}\big)^2\bigg)^2 + 2\bigg(\sum_{k=1}^{m} \big(B_{t-s}^{(k)}\big)^2\bigg)\big(B_{t-s}^{(m+1)}\big)^2 + \big(B_{t-s}^{(m+1)}\big)^4\Bigg].
\end{aligned}$$
Since coordinates of B_t are independent, we apply the induction hypothesis to derive that
$$\begin{aligned}
\mathbb{E}_0[|B_{t-s}|^4] &= m(m+2)|t-s|^2 + 2m(t-s)(t-s) + 3|t-s|^2 \\
&= (m+1)(m+3)|t-s|^2, \qquad (6)
\end{aligned}$$
where (2.2.10) in [2] is used in the second to last equality. This completes the induction step.

We can conclude that (5) holds for all $n \in \mathbb{N}$ by the principles of mathematical induction.

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2 Problem 2.16

Proof. Suppose $\widehat{B}_t := \frac{1}{c} B_{c^2 t}$ for $c > 0$. The continuity of sample path follows immediately from that of $(B_t)_{t \geqslant 0}$. To show $(\widehat{B}_t)_{t \geqslant 0}$ is a one-dimensional Brownian motion starting at x, it suffices to verify that it is a Gaussian process with finite dimensional distribution specified by the p.d.f.
$$\mathbb{P}_x(\widehat{B}_{t_1} \in \mathrm{d}x_1, \ldots, \widehat{B}_{t_k} \in \mathrm{d}x_k) = p_t(x, x_1) p_{t_2 - t_1}(x_1, x_2) \cdots p_{t_k - t_{k-1}}(x_{k-1}, x_k)\, \mathrm{d}x_1 \cdots \mathrm{d}x_k$$
for any $0 < t_1 < \cdots < t_k < \infty$ (Note that the notation that I use for transition probability density is different from what's given in [2] which should cause no confusion), where the transition probability density $p_t(x, y)$ is
$$p_t(x, y) = \frac{1}{(2\pi t)^{1/2}} \exp\bigg(-\frac{(x-y)^2}{2t}\bigg) \quad \text{for all } x, y \in \mathbb{R} \text{ and } t \in \mathbb{R}_{\geqslant 0}.$$
If $x_0 = x$ and $t_0 = 0$, we find that
$$\begin{aligned}
\mathbb{P}_x\big(B_{c^2 t_1} \in \mathrm{d}x_1, \ldots, B_{c^2 t_k} \in \mathrm{d}x_k\big) &= \prod_{i=1}^{k} \frac{1}{(2\pi c^2 (t_i - t_{i-1}))^{1/2}} \exp\bigg(-\frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})}\bigg)\, \mathrm{d}x_1 \cdots \mathrm{d}x_k \\
&= c^{-k} \prod_{i=1}^{k} \frac{1}{(2\pi (t_i - t_{i-1}))^{1/2}} \exp\bigg(-\frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})}\bigg)\, \mathrm{d}x_1 \cdots \mathrm{d}x_k \\
&= c^{-k} \prod_{i=1}^{k} p_{t_i - t_{i-1}}(x_{i-1}, x_i)\, \mathrm{d}x_1 \cdots \mathrm{d}x_k.
\end{aligned}$$

Therefore,

$$\mathbb{P}_x\big(cB_{c^2t_1} \in \mathrm{d}x_1, \ldots, cB_{c^2t_k} \in \mathrm{d}x_k\big) = \prod_{i=1}^{k} p_{t_i - t_{i-1}}(x_{i-1}, x_i)\,\mathrm{d}x_1 \cdots \mathrm{d}x_k.$$

This completes the proof. ∎

3 Problem 2.17

Proof.

1. Let $\mathcal{P}_n = \{t_0, t_1, \ldots, t_{n-1}, t_n\}$ with $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = t$ be a partition of $[0, t]$ and $\|\mathcal{P}_n\| = \max_{1 \leqslant k \leqslant n} |t_k - t_{k-1}|$. Note that

$$\mathbb{E}_x\left[\sum_{k=1}^{n}(B_{t_k} - B_{t_{k-1}})^2\right] = \sum_{k=1}^{n}(t_k - t_{k-1}) = t_n - t_0 = t.$$

Since one-dimensional Brownian motion has independent increments, then

$$\begin{aligned}\mathbb{E}\left[\left(\sum_{k=1}^{n}(B_{t_k} - B_{t_{k-1}})^2 - t\right)^2\right] &= \mathrm{Var}\left[\sum_{k=1}^{n}(B_{t_k} - B_{t_{k-1}})^2\right] \\ &= \sum_{k=1}^{n}\mathrm{Var}\left[(B_{t_k} - B_{t_{k-1}})^2\right] \\ &= 2\sum_{k=1}^{n}|t_k - t_{k-1}|^2,\end{aligned}$$

where the last equality follows from (5).

Note that

$$\sum_{k=1}^{n}|t_k - t_{k-1}|^2 \leqslant \|\mathcal{P}_n\|\sum_{k=1}^{n}|t_k - t_{k-1}| = t\|\mathcal{P}_n\|.$$

So $\sum_{k=1}^{n}(B_{t_k} - B_{t_{k-1}})^2 \xrightarrow{L^2} t$ as $\|\mathcal{P}_n\| \to 0$. An application of Chebyshev's inequality yields that $\sum_{k=1}^{n}(B_{t_k} - B_{t_{k-1}})^2 \xrightarrow{\mathrm{P}} t$ as $\|\mathcal{P}_n\| \to 0$. Since convergence in probability implies convergence a.s. along a subsequence, the quadratic variation $\langle B\rangle_t = t$ a.s..

2 We inherit the notations used in part (a). Hence

$$\sum_{k=1}^{n}|B_{t_k} - B_{t_{k-1}}|^2 \leqslant \sup_{1 \leqslant k \leqslant n}|B_{t_k} - B_{t_{k-1}}|\sum_{k=1}^{n}|B_{t_k} - B_{t_{k-1}}|.$$

The supremum on the right-hand side of the equation above tends to 0 by the continuity of sample paths, while the left-hand side tends to t a.s. as $\|\mathcal{P}_n\| \to 0$. It follows that $\sum_{k=1}^{n}|B_{t_k} - B_{t_{k-1}}|$ tends to $+\infty$ a.s. as $\|\mathcal{P}_n\| \to 0$. Therefore, the total variation on $[0, t]$

$$V_0^t(B) = \sup\left\{\sum_{k=1}^{n}|B_{t_k} - B_{t_{k-1}}| : \mathcal{P}_n = \{t_0, t_1, \ldots, t_n\} \text{ is a partition of } [0, t]\right\}$$

must be $+\infty$ a.s. ∎

References

[1] Chung, Kai Lai. A course in probability theory. Academic press, 2001.

[2] Oksendal, Bernt. Stochastic differential equations: an introduction with applications. Springer Science & Business Media, 2013. APA