2.8. (a), (b), (c), follow the direction.

(d). $n=1$, $\mathbb{E}[B_t^4] = 3t^2 = n(n+2)t^2$

$n=2$ $\mathbb{E}[|(B_{t,1}, B_{t,2})|^4] = \mathbb{E}[(B_{t_1}^2 + B_{t_2}^2)^2] = \mathbb{E}[B_{t_1}^4 + B_{t_2}^4 + 2B_{t_1}^2 B_{t_2}^2]$

$= 3t^2 + 3t^2 + 2t^2 = 8t^2 = n(n+2)t^2$

$n \geq 2$: $\mathbb{E}[|(B_{t_1}, \cdots B_{t_n})|^4] = \mathbb{E}[(B_{t_1}^2 + \sum_{i=2}^n B_{t_i}^2)^2]$

$= \mathbb{E}[B_{t_1}^4 + 2B_{t_1}^2 \sum_{i=2}^n B_{t_i}^2 + (\sum_{i=2}^n B_{t_i}^2)^2]$

$= 3t^2 + 2t(n-1)t + \underbrace{\mathbb{E}[(\sum_{i=2}^n B_{t_i}^2)^2]}_{(n-1)(n+1)t^2}$

$= [3 + 2(n-1) + n^2 - 1] t^2 = (n^2 + 2n) t^2$.

2.16. By GP definition; verify the covariance.

2.17. Show that Bm has unbdd TV a.s. from $\mathbb{E}[\sum_{t_k \leq t} (\Delta B_{t_k})^2 - t] = 2\sum_{t_k < t} (\Delta t_k)^2 \leq 2t\Delta \to 0$ ($\Delta = \max_k \Delta t_k$)

$\Rightarrow Y_t^\Delta := \sum_{t_k \leq t} |\Delta B_{t_k}(\omega)|^2 \to t$ in $L^2(\mathbb{P})$. (*)

Proof: ① Note that (a) implies $Y_t^\Delta := \sum_{t_k \leq t} |\Delta B_{t_k}(\omega)|^2 \to t$ a.s.

(You can also use $\langle B \rangle_t = t$ a.s. here).

② Let $V_t(\omega) = \lim_{\Delta t_k \to 0} \sum_{t_k \leq t} |B_{t_{k+1}}(\omega) - B_{t_k}(\omega)| < \infty$ for some ω.

Then, noting that $\sup_k |\Delta B_{t_k}| \xrightarrow[\Delta \downarrow 0]{} 0$ b/c $B_\cdot(\omega)$ is cts,

we have, as $\Delta \downarrow 0$,

$$Y_t^\Delta = \sum_{t_k \leq t} |\Delta B_{t_k}(\omega)|^2 \leq \underbrace{\sup_\Delta |\Delta B_{t_k}|}_{\to 0} \underbrace{\sum_{t_k \leq t} |\Delta B_{t_k}|}_{\to V_t(\omega)} \to 0.$$

$\xrightarrow{① \& ②}$ $\mathbb{P}(\omega: V_t(\omega) < \infty) = 0$.

← If $X_n \to X$ in $L^2(\mathbb{P})$

then by Chebyshev's Ineq:

$\mathbb{P}(|X_n - X| > \varepsilon) \leq \varepsilon^{-2}\mathbb{E}[|X_n - X|^2] \xrightarrow{n \uparrow \infty} 0$

① i.e. $X_n \to X$ in prob.

② $\Delta_k = 2^{-k} \cdot 2^{-k}$, $\varepsilon_k = 2^{-k/2}$,

$\mathbb{P}(\underbrace{|Y_t^{\Delta_k} - t| > 2^{-k}}_{A_k}) \leq 2^{+k} 2t 2^{-2k} = 2^{-k} \cdot 2t$

$\sum_{k=1}^\infty 2^{-k} 2t < \infty$

Borel-Cantelli $\Rightarrow$

$\mathbb{P}(\bigcap_{m=1}^\infty \bigcup_{k=m}^\infty A_k) = 0$

$= \mathbb{P}(\lim_{k \uparrow \infty} |Y_t^{\Delta_k} - t| > 0)$

i.e. $Y_t^{\Delta_k} \to t$ a.s.