110.653: Introduction to SDE

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1 Problem 8.2

Proof. The initial value problem that we are interested in is

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{2}\beta^2 x^2 \frac{\partial^2 u}{\partial x^2} + \alpha x \frac{\partial u}{\partial x} & \text{for } t > 0, x \in \mathbb{R}; \\ u(0,x) = f(x) & \text{for } f \in C_0^2(\mathbb{R}). \end{cases}$$

From Theorem 7.3.3 in [2], the one dimensional Itô diffusion satisfying

$$\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}B_t$$

has infinitesimal generator

$$Af(x) = b(x)\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2(x)\frac{\partial^2 f}{\partial x^2}.$$

Comparing the coefficients, one finds that $b(x) = \alpha x$ and $\sigma(x) = \beta x$, hence the corresponding SDE becomes

$$\mathrm{d}X_t = \alpha X_t\,\mathrm{d}t + \beta X_t\,\mathrm{d}B_t. \tag{1}$$

In Example 5.1.1 of [2] by applying Itô's formula to the natural logarithm function, we know the solution to (1) is the geometric Brownian motion given by

$$X_t = X_0 \exp((\alpha - \beta^2/2)t + \beta B_t).$$

The Kolmogorov's backward equation (Theorem 8.1.1 in [2]) yields

$$u(t,x) = \mathbb{E}_x[f(x\exp((\alpha - \beta^2/2)t + \beta B_t))].$$

Writing u in terms of the Gaussian density, the desired expression

$$u(t,x) = \frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} f(x\exp((\alpha - \beta^2/2)t + \beta y))\exp\left(-\frac{y^2}{2t}\right)\mathrm{d}y$$

follows. □

2 Problem 8.6

The initial value problem of interest is given by

$$\begin{cases} \frac{\partial u}{\partial t} = -\rho u + \alpha x \frac{\partial u}{\partial x} + \frac{1}{2}\beta^2 x^2 \frac{\partial^2 u}{\partial x^2} & \text{for } t > 0, x \in \mathbb{R}; \\ u(0,x) = f(x) & \text{for } x \in \mathbb{R}, \end{cases}$$

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where $\rho > 0$, α, β and $K > 0$ are constants and

$$f(x) = (x - K)^+.$$

Note that $f \notin C_0^2(\mathbb{R})$, so the Feynman-Kac formula given in [2] is not applicable. But the behavior of f is not that terrible, it is continuous with polynomial growth. We can still apply the version of Feynmann-Kac formula given in Theorem 7.6 of [1] with time reversal. Now we need to verify that

$$\max_{0 \le t \le T} |v(t,x)| \le M(1 + |x|^{2\mu})$$

for some $M > 0$ and $\mu \ge 1$. It can be done by checking the local version of the sufficient conditions given in Remark 7.8 of [1]. With $(X_t)_{t \ge 0}$ satisfying (1) and $q \equiv \rho$, we apply the Feyman-Kac formula to derive that

$$u(t,x) = \mathbb{E}_x[\exp(-\rho t)(x \exp((\alpha - \beta^2/2)t + \beta B_t) - K)^+].$$

Writing the equation above in terms of the Gaussian density, we have

$$u(t,x) = \frac{e^{-\rho t}}{\sqrt{2\pi t}} \int_{\mathbb{R}} (x \exp((\alpha - \beta^2/2)t + \beta y) - K)^+ \exp\left(-\frac{y^2}{2t}\right) \mathrm{d}y$$

as required.

References

[1] Karatzas, I. and Shreve, E. S. (1991). *Brownian Motion and Stochastic Calculus*, 2nd ed. Graduate Texts in Mathematics (113). Springer, New York.

[2] Øksendal, B. (2013). *Stochastic Differential Equations*, 6th ed. Springer-Verlag, Berlin Heidelberg.