110.653: Introduction to SDE

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1 Problem 6.1

Proof. In Theorem 6.2.8 of [1], we take $F = C \equiv 0$ and $D \equiv 1$. Hence the Riccati equation satisfied by the error term $S(t)$ becomes

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -G(t)^2 S(t)^2 \quad \text{with initial condition } S(0).$$

The variables in the ODE above are well-separated, hence we have

$$\frac{1}{S(t)} = C + \int_0^t G(s)^2 \,\mathrm{d}s \tag{1}$$

for some constant $C \in \mathbb{R}$. Evaluating (1) at $t = 0$, then $C = 1/S(0)$. So the solution to the ODE is

$$S(t) = \frac{1}{\frac{1}{S(0)} + \int_0^t G^2(s)\,\mathrm{d}s}.$$

To show the second assertion, suppose $0 < p \le 1/2$, then

$$\int_0^\infty \frac{1}{(1+s)^{2p}}\,\mathrm{d}s = \begin{cases} \lim_{s\to\infty} \ln(1+s) & \text{if } p = 1/2; \\ \lim_{s\to\infty} \frac{1}{1-2p}(1+s)^{1-2p} - \frac{1}{1-2p} & \text{if } 0 < p < 1/2. \end{cases}$$

Clearly, the integral tends to $+\infty$ in this case. Conversely, suppose $p > 1/2$, one has

$$\infty = \int_0^\infty G(s)^2\,\mathrm{d}s = \int_0^\infty \frac{1}{(1+s)^{2p}}\,\mathrm{d}s = \lim_{s\to\infty} \frac{1}{1-2p}(1+s)^{1-2p} - \frac{1}{1-2p} = -\frac{1}{1-2p} < \infty,$$

which is a contradiction. Hence $0 < p \le 1/2$. □

2 Problem 6.2

Proof. **(a)**. Taking $C \equiv 0$ in Theorem 6.2.8 in [1], then the Riccati equation for $S(t)$ becomes

$$\frac{\mathrm{d}S}{\mathrm{d}t} = 2F(t)S(t) - \frac{G(t)^2}{D(t)^2} S(t)^2. \tag{2}$$

If $R := 1/S$, then, by the chain rule,

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -\frac{1}{R(t)^2}\frac{\mathrm{d}R}{\mathrm{d}t}.$$

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Plugging these expressions back into (2), the ODE involving R is

$$\begin{aligned} -\frac{1}{R(t)^2}\frac{\mathrm{d}R}{\mathrm{d}t} &= 2F(t)\frac{1}{R(t)} - \frac{G(t)^2}{D(t)^2}\frac{1}{R(t)^2} \\ \frac{\mathrm{d}R}{\mathrm{d}t} &= -2F(t)R(t) + \frac{G(t)^2}{D(t)^2}. \end{aligned} \tag{3}$$

The initial condition $R(0) = 1/S(0)$ is immediate.

(b). Multiplying both sides of (3) by

$$\exp\left(2\int_0^t F(s)\,\mathrm{d}s\right) \tag{4}$$

and then rearranging the terms,

$$\begin{aligned} \exp\left(2\int_0^t F(s)\,\mathrm{d}s\right)R'(t) + 2F(t)R(t)\exp\left(2\int_0^t F(s)\,\mathrm{d}s\right) &= \frac{G(t)^2}{D(t)^2}\exp\left(2\int_0^t F(s)\,\mathrm{d}s\right) \\ \frac{\mathrm{d}}{\mathrm{d}t}\left[\exp\left(2\int_0^t F(s)\,\mathrm{d}s\right)R(t)\right] &= \frac{G(t)^2}{D(t)^2}\exp\left(2\int_0^t F(s)\,\mathrm{d}s\right) \\ \exp\left(2\int_0^t F(s)\,\mathrm{d}s\right)R(t) - R(0) &= \int_0^t \frac{G(u)^2}{D(u)^2}\exp\left(2\int_0^u F(s)\,\mathrm{d}s\right)\mathrm{d}u. \end{aligned}$$

Dividing both sides by the factor in (4), we can conclude that

$$R(t) = \exp\left(-2\int_0^t F(s)\,\mathrm{d}s\right)R(0) + \int_0^t \exp\left(-2\int_u^t F(s)\,\mathrm{d}s\right)\frac{G(u)^2}{D(u)^2}\,\mathrm{d}u.$$

This completes the proof. □

References

[1] ØKSENDAL, B. (2013). *Stochastic Differential Equations*, 6th ed. Springer-Verlag, Berlin Heidelberg.