Stochastic ODEs HW6

Joshua Agterberg

Problem 6.1

By Theorem 6.2.8, we see that $S(t)$ satisfies the Ricatti equation

$$\frac{dS}{dt} = -G^2(t)S^2(t) \tag{1}$$

where we have used the fact that $dX_t = 0$ and $dZ_t = G(t)X_t dt + dV_t$ to simplify the expression (i.e. $F = C = 0$; $D(t) \equiv 1$ in the statement of the theorem). The non-zero solution of equation (1) is (because $(\frac{1}{S})' = -\frac{S'}{S^2} = G^2$)

$$S(t) = -\left[\frac{1}{S_0} + \int_0^t G^2(s)ds\right]^{-1}.$$

The second half of the problem is immediate, as the integral

$$\int_0^\infty \frac{1}{(1+s)^{2p}}ds$$

is only integrable when $p > \frac{1}{2}$.

Problem 6.2

(a) Again by Theorem 6.2.8, we have that

$$\frac{dS}{dt} = 2F(t)S(t) - \frac{G^2(t)}{D^2(t)}S^2(t).$$

Since $R(t) = \frac{1}{S(t)}$, we have that

$$\begin{aligned} R'(t) &= \frac{d}{dt}\frac{1}{S(t)} = -\frac{S'(t)}{S^2(t)} = \frac{-1}{S^2(t)}\left(2F(t)S(t) - \frac{G^2(t)}{D^2(t)}S^2(t)\right) \\ &= -1\frac{2F(t)}{S(t)} + \frac{G^2(t)}{D^2(t)} = -2F(t)R(t) + \frac{G^2(t)}{D^2(t)} \end{aligned}$$

as desired. The initial condition is immediate.

(b) From above we see that $R(t)$ satisfies a non-homogeneous first-order ODE. Let

$$\eta(t) := -2\int_0^t F(s)ds.$$

The general solution to such an equation is given by

$$\begin{aligned} R(t) &= R(0)e^{\eta(t)} + e^{\eta(t)}\int_0^t \frac{G^2}{D^2}(s)e^{-\eta(s)}ds \\ &= R(0)\exp\left(-2\int_0^t F(s)ds\right) + \exp\left(-2\int_0^t F(s)ds\right)\int_0^t \frac{G^2}{D^2}(s)\exp\left(2\int_0^s F(s)ds\right)ds \\ &= R(0)\exp\left(-2\int_0^t F(s)ds\right) + \int_0^t \frac{G^2}{D^2}(s)\exp\left(-2\int_s^t F(s)ds\right)ds. \end{aligned}$$

Noting that $R(t) = S(t)^{-1}$ gives the desired result.