Stochastic ODEs HW5

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Problem 1

We assume without loss of generality that $W_0 = 0$ (or otherwise replace W_t by $W_t - W_0$).

First, observe that by Ito's formula applied to $a(X_t)$, it holds that

$$\begin{aligned} da(X_t) &= a'(X_t)dX_t + \frac{1}{2}a''(X_t)(dX_t)^2 \\ &= a'(X_t)\Big[a(X_t)dt + b(X_t)dW_t\Big] + \frac{1}{2}a''(X_t)b^2(X_t)dt, \end{aligned} \tag{1}$$

where the standard convention $(dW_t)^2 = dt$ has been used implicitly. Consequently, we have the Ito-Taylor expansion

$$\begin{aligned} [a(X_s) - a(X_0)] &= \int_0^s da(X_r) \\ &= \int_0^s \Big[a'(X_r)\Big[a(X_r)dr + b(X_r)dW_r\Big] + \frac{1}{2}a''(X_r)b^2(X_r)dr\Big] \\ &= \int_0^s a'(X_r)a(X_r)dr + \int_0^s a'(X_r)b(X_r)dW_r + \int_0^s \frac{1}{2}a''(X_r)b^2(X_r)dr. \end{aligned}$$

Therefore, we obtain that

$$\begin{aligned} \int_0^t (a(X_s) - a(X_0))ds &= \int_0^t \Big\{\int_0^s a'(X_r)a(X_r)dr + \int_0^s a'(X_r)b(X_r)dW_r + \int_0^s \frac{1}{2}a''(X_r)b^2(X_r)dr\Big\}ds \\ &= \int_0^t\int_0^s a'(X_r)a(X_r)drds + \int_0^t\int_0^s a'(X_r)b(X_r)dW_r ds \\ &\quad + \int_0^t\int_0^s \frac{1}{2}a''(X_r)b^2(X_r)drds. \end{aligned} \tag{2}$$

By applying the analogous argument for $db(X_t)$ as in (1), we have

$$\begin{aligned} \int_0^t \big(b(X_s) - b(X_0)\big)dW_s &= \int_0^t\int_0^s \Big\{b'(X_r)\Big[a(X_r)dr + b(X_r)dW_r\Big] + \frac{1}{2}b''(X_r)b^2(X_r)dr\Big\}dW_s \\ &= \int_0^t\int_0^s b'(X_r)a(X_r)drdW_s + \int_0^t\int_0^s b'(X_r)b(X_r)dW_r dW_s \\ &\quad + \int_0^t\int_0^s \frac{1}{2}b''(X_r)b^2(X_r)drdW_s. \end{aligned} \tag{3}$$

Therefore, we have that

$$\begin{aligned} X_t &= X_0 + \int_0^t a(X_0)ds + \int_0^t b(X_0)dW_s + \int_0^t \big[a(X_s) - a(X_0)\big]ds + \int_0^t \big[b(X_s) - b(X_0)\big]dW_s \\ &= X_0 + ta(X_0) + W_t b(X_0) + \int_0^t \big[a(X_s) - a(X_0)\big]ds + \int_0^t \big[b(X_s) - b(X_0)\big]dW_s \\ &= X_0 + ta(X_0) + W_t b(X_0) \\ &\quad + \int_0^t\int_0^s a'(X_r)a(X_r)drds + \int_0^t\int_0^s a'(X_r)b(X_r)dW_r ds + \int_0^t\int_0^s \frac{1}{2}a''(X_r)b^2(X_r)drds \\ &\quad + \int_0^t\int_0^s b'(X_r)a(X_r)drdW_s + \int_0^t\int_0^s b'(X_r)b(X_r)dW_r dW_s + \int_0^t\int_0^s \frac{1}{2}b''(X_r)b^2(X_r)drdW_s, \end{aligned}$$

where in the final equality we have plugged in (2) and (3). Rearranging similar terms we obtain

$$
\begin{aligned}
X_t = X_0 &+ ta(X_0) + W_t b(X_0) \\
&+ \int_0^t \int_0^s \Big(a'(X_r)a(X_r) + \frac{1}{2}a''(X_r)b^2(X_r)\Big) dr ds + \int_0^t \int_0^s a'(X_r)b(X_r)dW_r ds \\
&+ \int_0^t \int_0^s \Big(b'(X_r)a(X_r) + \frac{1}{2}b''(X_r)b^2(X_r)\Big) dr dW_s + \int_0^t \int_0^s b'(X_r)b(X_r)dW_r dW_s. \quad (4)
\end{aligned}
$$

This is where we would stop for the schemes studied in class. However, for the 1.5-order scheme, we also need to further expand the double stochastic integral. Note that

$$
\begin{aligned}
d\Big(b'b\Big) &= (b'b)'(X_t)dX_t + \frac{1}{2}(b'b)''(X_t)(dX_t)^2 \\
&= (b'b)'a(X_t)dt + (b'b)'(X_t)b(X_t)dW_t + \frac{1}{2}(b'b)''(X_t)b^2(X_t)dt.
\end{aligned}
$$

Therefore, we expand out

$$
\begin{aligned}
\int_0^t \int_0^s b'(X_r)b(X_r)dW_r dW_s &= \int_0^t \int_0^s \big(b'(X_r)b(X_r) - b'(X_0)b(X_0)\big) dW_r dW_s + \int_0^t \int_0^s b'(X_0)b(X_0)dW_r dW_s \\
&= \int_0^t \int_0^s \int_0^r \Big((b'b)'(X_t)a(X_t)d_q + (b'b)'(X_t)b(X_t)dW_q \\
&\qquad + \frac{1}{2}(b'b)''(X_t)b^2(X_t)dq \Big) dW_r dW_s \\
&\quad + \int_0^t \int_0^s b'(X_0)b(X_0)dW_r dW_s \\
&= \int_0^t \int_0^s \int_0^r (b'b)'(X_t)a(X_t)dq dW_r dW_s + \int_0^t \int_0^s \int_0^r (b'b)'(X_t)b(X_t)dW_q dW_r dW_s \\
&\quad + \int_0^t \int_0^s b'(X_0)b(X_0)dW_r dW_s \\
&\quad + \int_0^t \int_0^s \int_0^r \frac{1}{2}(b'b)''(X_t)b^2(X_t)dq dW_r dW_s \\
&= \int_0^t \int_0^s \int_0^r (b'b)'(X_t)b(X_t)dW_q dW_r dW_s + \int_0^t \int_0^s b'(X_0)b(X_0)dW_r dW_s + o(t^{3/2}) \\
&= \int_0^t \int_0^s \int_0^r (b'b)'(X_t)b(X_t)dW_q dW_r dW_s + b'(X_0)b(X_0)\int_0^t W_s dW_s + o(t^{3/2}),
\end{aligned}
$$

where the notation $o(\cdot)$ is taken to mean in probability as $t \to 0$. Therefore, plugging this back into equation (4), we obtain

$$
\begin{aligned}
X_t = X_0 &+ ta(X_0) + W_t b(X_0) \\
&+ \int_0^t \int_0^s \Big(a'(X_r)a(X_r) + \frac{1}{2}a''(X_r)b^2(X_r)\Big) dr ds + \int_0^t \int_0^s a'(X_r)b(X_r)dW_r ds \\
&+ \int_0^t \int_0^s \Big(b'(X_r)a(X_r) + \frac{1}{2}b''(X_r)b^2(X_r)\Big) dr dW_s \\
&+ \int_0^t \int_0^s \int_0^r (b'b)'(X_t)b(X_t)dW_q dW_r dW_s + b'(X_0)b(X_0)\int_0^t W_s dW_s + o(t^{3/2}). \quad (5)
\end{aligned}
$$

We now approximate $a'(X_r), a(X_r), b(X_r), b'(X_r)$ and $b''(X_r)$ by their leftpoint evaluations at

X_0 respectively. Then for small t,

$$
\begin{aligned}
X_t &= X_0 + ta(X_0) + W_t b(X_0) \\
&\quad + \left(a'(X_0)a(X_0) + \frac{1}{2}a''(X_0)b^2(X_0)\right)\int_0^t\int_0^s drds + a'(X_0)b(X_0)\int_0^t\int_0^s dW_r ds \\
&\quad + \left(b'(X_0)a(X_0) + \frac{1}{2}b''(X_0)b^2(X_0)\right)\int_0^t\int_0^s dr dW_s \\
&\quad + b'(X_0)b(X_0)\int_0^t W_s dW_s \\
&\quad + (b'b)'(X_0)b(X_0)\int_0^t\int_0^s\int_0^r dW_q dW_r dW_s \\
&\quad + b'(X_0)b(X_0)\int_0^t W_s dW_s \\
&= X_0 + ta(X_0) + W_t b(X_0) \\
&\quad + \frac{t^2}{2}\left(a'(X_0)a(X_0) + \frac{1}{2}a''(X_0)b^2(X_0)\right) + a'(X_0)b(X_0)\int_0^t W_s ds \\
&\quad + \left(b'(X_0)a(X_0) + \frac{1}{2}b''(X_0)b^2(X_0)\right)\int_0^t s dW_s \\
&\quad + (b'b)'(X_0)b(X_0)\int_0^t\int_0^s W_r dW_r dW_s \\
&\quad + b'(X_0)b(X_0)\int_0^t W_s dW_s \\
&= X_0 + ta(X_0) + W_t b(X_0) \\
&\quad + \frac{t^2}{2}\left(a'(X_0)a(X_0) + \frac{1}{2}a''(X_0)b^2(X_0)\right) + a'(X_0)b(X_0)\int_0^t W_s ds \\
&\quad + \left(b'(X_0)a(X_0) + \frac{1}{2}b''(X_0)b^2(X_0)\right)\left(tW_t - \int_0^t W_s ds\right) + (b'b)'(X_0)b(X_0)\int_0^t\int_0^s W_r dW_r dW_s \\
&\quad + b'(X_0)b(X_0)\int_0^t W_s dW_s \\
&= X_0 + ta(X_0) + W_t b(X_0) \\
&\quad + \frac{t^2}{2}\left(a'(X_0)a(X_0) + \frac{1}{2}a''(X_0)b^2(X_0)\right) + a'(X_0)b(X_0)\int_0^t W_s ds \\
&\quad + \left(b'(X_0)a(X_0) + \frac{1}{2}b''(X_0)b^2(X_0)\right)\left(tW_t - \int_0^t W_s ds\right) + \frac{1}{2}(b'b)'(X_0)b(X_0)\int_0^t (W_s^2 - s) dW_s \\
&\quad + \frac{1}{2}b'(X_0)b(X_0)(W_t^2 - t). \qquad (6)
\end{aligned}
$$

Note that we used here the fact that $\int_0^t W_s dW_s = \frac{1}{2}(W_t^2 - t)$. We now calculte the integral $\int_0^t (W_s^2 - s) dW_s$. By Exercises 3.1 and 3.2,

$$
\begin{aligned}
\int_0^t W_s^2 dW_s - \int_0^t s dW_s &= \frac{1}{3}W_t^3 - \int_0^t W_s ds - \int_0^t s dW_s \\
&= \frac{1}{3}W_t^3 - tW_t.
\end{aligned}
$$

Plugging this back into (6), we obtain

$$
\begin{aligned}
X_t = X_0 + ta(X_0) + W_t b(X_0) &+ \frac{t^2}{2}\left(a'(X_0)a(X_0) + \frac{1}{2}a''(X_0)b^2(X_0)\right) \\
&+ a'(X_0)b(X_0)\int_0^t W_s ds + \left(b'(X_0)a(X_0) + \frac{1}{2}b''(X_0)b^2(X_0)\right)\left(tW_t - \int_0^t W_s ds\right) \\
&+ \frac{1}{6}(b'b)'(X_0)b(X_0)\left(W_t^3 - 3tW_t\right) + \frac{1}{2}b'(X_0)b(X_0)(W_t^2 - t).
\end{aligned}
$$

Note that the only random variables appearing are W_t and $\int_0^t W_s ds$. Let $Z_t = \int_0^t W_s ds$. Note that both W_t and Z_t are Gaussian random variables with mean zero, so it suffices to calculate their second moments and covariance. Clearly $W_t \sim N(0,t)$. Furthermore,

$$\begin{aligned}
\mathbb{E}\Big(\int_0^t W_s ds\Big)^2 &= \mathbb{E}\Big(tW_t - \int_0^t s dW_s\Big)^2 \\
&= \mathbb{E}(t^2 W_t^2) - \mathbb{E}2tW_t \int_0^t s dW_s + \mathbb{E}\Big(\int_0^t s dW_s\Big)^2 \\
&= t^3 - 2t\mathbb{E}W_t(tW_t - \int_0^t W_s ds) + \mathbb{E}\Big(\int_0^t s dW_s\Big)^2 \\
&= t^3 - 2t^3 + 2t\mathbb{E}W_t \int_0^t W_s ds + \mathbb{E}\int_0^t s^2 ds \\
&= t^3 - 2t^3 + 2t\int_0^t \min(t,s) ds + \frac{1}{3}t^3 \\
&= t^3 - 2t^3 + 2t\int_0^t s ds + \frac{1}{3}t^3 \\
&= \frac{1}{3}t^3.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\mathbb{E}W_t Z_t = \mathbb{E}W_t \int_0^t W_s ds &= \int_0^t \mathbb{E}W_t W_s ds \\
&= \int_0^t \min(t,s) ds = \int_0^t s ds = \frac{1}{2}t^2.
\end{aligned}$$

Therefore, $Z_t \sim N(0, \frac{1}{3}t^3)$ and $W_t \sim N(0,t)$, and the two have covariance $\frac{1}{2}t^2$. To simulate this, we want independent Gaussians. Let ξ_1 and ξ_2 be independent $N(0,1)$ random variables. Note that for positive coefficients a_1, a_2, and a_3,

$$\mathbb{E}a_1\xi_1 a_2(a_3\xi_1 + a_4\xi_2) = a_1 a_2 a_3.$$

Therefore, if $a_1\xi_1 = W_t$ in distribution, we must have $a_1 = \sqrt{t}$. Moreover, by matching expectations we need that

$$\begin{aligned}
t^{1/2}a_2 a_3 &= \frac{1}{2}t^2; \\
a_2^2 a_3^2 + a_2^2 a_4^2 &= \frac{1}{3}t^3.
\end{aligned}$$

This is a system with several solutions. Therefore, we can arbitrarily select $a_3 = 1$, to see that $a_2 = \frac{1}{2}t^{3/2}$ and $a_4 = \frac{1}{\sqrt{3}}$. Consequently, if we set

$$\Delta Z_n := a_2(a_3\xi_1 + a_4\xi_2) = \frac{1}{2}t^{3/2}(\xi_1 + \frac{1}{\sqrt{3}}\xi_2)$$

and $\Delta W_n = a_1\xi_1 = \sqrt{t}\xi_1$, we see that

$$\mathbb{E}\Delta Z_n \Delta W_n = \frac{1}{2}t^2.$$

With this, we arrive at the 1.5-order scheme:

$$\begin{aligned}
X_{n+1} = X_n + \Delta t a(X_n) + \Delta W_n b(X_n) + \frac{(\Delta t)^2}{2}\Big(a'(X_0)a(X_0) + \frac{1}{2}a''(X_n)b^2(X_n)\Big) \\
+ a'(X_0)b(X_0)\Delta Z_n + \Big(b'(X_n)a(X_n) + \frac{1}{2}b''(X_n)b^2(X_n)\Big)\Big(\Delta t\Delta W_n - \Delta Z_n\Big) \\
+ \frac{1}{6}(b'b)'(X_0)b(X_0)\Big((\Delta W_n)^3 - 3\Delta t\Delta W_n\Big) + \frac{1}{2}b'(X_0)b(X_0)((\Delta W_n)^2 - \Delta t).
\end{aligned}$$

Problem 2

(a) First, we have that

$$\begin{aligned} Y_n &= (1+\lambda\delta)Y_{n-1} + \sigma\sqrt{\delta}\xi_{n-1} \\ &= (1+\lambda\delta)^2 Y_{n-2} + (1+\lambda\delta)\sigma\sqrt{\delta}\xi_{n-2} + \sigma\sqrt{\delta}\xi_{n-1}. \end{aligned}$$

Therefore, doing this n times yields

$$Y_n = (1+\lambda\delta)^n Y_0 + \sigma\sqrt{\delta}\sum_{k=0}^{n-1}(1+\lambda\delta)^k \xi_{n-k-1}.$$

Taking second moments, by independence of ξ_k's, we have

$$\begin{aligned} \mathbb{E}Y_n^2 &= (1+\lambda\delta)^{2n}\mathbb{E}Y_0^2 + \sigma^2\delta\sum_{k=0}^{n-1}(1+\lambda\delta)^{2k} \\ &= \sigma^2\delta\sum_{k=0}^{n-1}(1+\lambda\delta)^{2k}. \end{aligned}$$

Recall that $\lambda < 0$. If $\delta < 0$, then $1+\lambda\delta > 1$, meaning that this sum does not converge. If $\delta > 0$, then this sum converges as long as $\delta < -\frac{2}{\lambda}$. Therefore, the full range of convergence is given by $0 < \delta < \frac{2}{|\lambda|}$. Though it should perhaps be noted that this expectation is finite for all n regardless of the choice of δ, though if $\delta \geq \frac{2}{|\lambda|}$ then this sum is of order n (as opposed to order 1 in the other case).

To calculate the limiting variance, we can sum the geometric series:

$$\begin{aligned} \lim \mathbb{E}Y_n^2 &= \sigma^2\delta\sum_{k=0}^{\infty}(1+\lambda\delta)^{2k} \\ &= \sigma^2\delta\left(\frac{1}{1-(1+\lambda\delta)^2}\right) \\ &= \sigma^2\delta\frac{1}{-2\lambda\delta-\lambda^2\delta^2} \\ &= \frac{\sigma^2}{-2\lambda-\lambda^2\delta}. \end{aligned}$$

To check this, we can also solve the fixed-point equation:

$$\begin{aligned} x &= (1+\lambda\delta)^2 x + \sigma^2\delta \\ \implies x &= \frac{\sigma^2\delta}{1-(1+\lambda\delta)^2} = \frac{\sigma^2}{-2\lambda-\lambda^2\delta}. \end{aligned}$$

To check this matches our intuition, as $\delta \to \frac{2}{|\lambda|}$, the time step increases, so the variance should increase, which it does. Similarly, $\delta \to 0$, the time step decreases, so the variance should decrease, which it does.

(b) Similar to part (a), we rearrange and expand out the recursion to obtain

$$\begin{aligned} Y_n &= (1-\lambda\delta)^{-1}Y_{n-1} + \frac{\sigma\sqrt{\delta}}{1-\lambda\delta}\xi_{n-1} \\ &= (1-\lambda\delta)^{-n}Y_0 + \frac{\sigma\sqrt{\delta}}{1-\lambda\delta}\sum_{k=0}^{n-1}(1-\lambda\delta)^{-k}\xi_{n-k-1} \\ &= \frac{\sigma\sqrt{\delta}}{1-\lambda\delta}\sum_{k=0}^{n-1}(1-\lambda\delta)^{-k}\xi_{n-k-1}. \end{aligned}$$

Just as in part (a), this sum converges when $|1 - \lambda\delta| > 1$, which means that we need $1 - \lambda\delta > 1$, which is equivalent to $\delta > 0$ since $\lambda < 1$. Similarly, we also need $\delta > 2/\lambda$, which is vacuous compared to $\delta > 0$. Therefore, we only need $\delta > 0$.

To calculate the limiting variance we sum up the geometric series as before:

$$\begin{aligned}
\lim \mathbb{E}Y_n^2 &= \frac{\sigma^2\delta}{(1-\lambda\delta)^2} \sum_{k=0}^{\infty} (1-\lambda\delta)^{-2k} \\
&= \frac{\sigma^2\delta}{(1-\lambda\delta)^2} \frac{1}{1 - \frac{1}{(1-\lambda\delta)^2}} \\
&= \frac{\sigma^2\delta}{(1-\lambda\delta)^2} \frac{(1-\lambda\delta)^2}{(1-\lambda\delta)^2 - 1} \\
&= \frac{\sigma^2\delta}{-2\lambda\delta + \lambda^2\delta^2} \\
&= \frac{\sigma^2}{-2\lambda + \lambda^2\delta}.
\end{aligned}$$

This is again an increasing function for $\delta > 0$, with lower bound $\frac{\sigma^2}{-2\lambda}$ as $\delta \to 0$. This matches with our intuition that the limiting variance should increase for larger time steps. We could also solve the fixed-point equation to check our work:

$$\begin{aligned}
x &= (1-\lambda\delta)^{-2}x + \frac{\sigma^2\delta}{(1-\lambda\delta)^2} \\
\implies x &= \frac{\sigma^2}{-2\lambda + \lambda^2\delta}.
\end{aligned}$$