national math festival

The mathematics of social choice

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Suppose Amy and Beto are running for office.
How do we decide who wins?

- ⋆ Everyone* votes.
- ⋆ Whichever candidates gets the most votes wins.

It makes sense that in a head-to-head race,
whoever receives the majority of the votes should win.

*or at least everyone who is able to vote

Suppose Amy, Andrew, Bernie, Beto, Cory, Elizabeth, Eric, Jay, Joe, John, John, Julián, Kamala, Kirsten, Marianne, Michael, Mike, Pete, Seth, Tim, Tulsi, Wayne are running for office.
How do we decide who wins?

- Everyone* votes.
- Whichever candidates gets the most votes wins?

This is the system used to determine the winner in most states' presidential primaries: whoever gets a plurality of votes wins.

*or at least everyone who is able to vote

With a large candidate pool

Amy, Andrew, Bernie, Beto, Cory, Elizabeth, Eric,
Jay, Joe, John, John, Julián, Kamala, Kirsten,
Marianne, Michael, Mike, Pete, Seth, Tim, Tulsi, Wayne

the plurality method can run into problems:

- ⋆ The winner might earn a very low proportion of the vote
  — with 22 candidates you may need only 5 percent to win!
- ⋆ The rankings can change dramatically if any candidate drops out.
- ⋆ If polling identifies likely front runners, voters may choose to vote strategically for a candidate who is not their true top choice.

Other voting systems can be used to determine the outcome of an election with more than two candidates.

- ⋆ In the plurality method, each voter votes for one candidate.
- ⋆ In the vote-for-two method, each voter votes for two candidates.
- ⋆ ...
- ⋆ In the anti-plurality method, each voter votes for all but one of the candidates, effectively casting a vote *against* their last choice.

As we shall discover, the voting system matters: the candidate who wins a plurality election, might not win with vote-for-two or with anti-plurality.

In the GPA method, each voter ranks all n candidates.

- ⋆ Their first choice earns $n–1$ points.
- ⋆ Their second choice earns $n–2$ points.
- ⋆ ...
- ⋆ Their second to last choice earns 1 point.
- ⋆ Their last choice earns 0 points.

The candidate with the most points —"the highest GPA" — wins.

In approval voting each voter may

- ⋆ vote for one candidate or
- ⋆ vote for two candidates or
- ⋆ ...
- ⋆ vote for all but one of the candidates

...and each voter gets to decide how many candidates to vote for!

Does the election outcome reflect the will of the people
…or the choice of the voting method?

Suppose there are nine voters and four candidates:

two voters prefer	Amy	>	Beto	>	Cory	>	Liz
two voters prefer	Amy	>	Liz	>	Cory	>	Beto
two voters prefer	Cory	>	Beto	>	Liz	>	Amy
three voters prefer	Liz	>	Beto	>	Cory	>	Amy

- ⋆ Amy wins plurality: Amy (4) > Liz (3) > Cory (2) > Beto (0)
- ⋆ Beto wins vote-for-two: Beto (7) > Liz (5) > Amy (4) > Cory (2)
- ⋆ Cory wins anti-plurality: Cory (9) > Beto (7) = Liz (7) > Amy (4)
- ⋆ Liz wins GPA: Liz (15) > Beto (14) > Cory (13) > Amy (12)

And any candidate could win with approval voting depending on how many candidates each voter chooses to vote for!

Multi-round voting systems simulate elections with runoffs after eliminating some candidates — but voters only cast their ballots once.

In ranked-choice voting each voter ranks all of the candidates:

- ⋆ If no candidate wins a majority of first-place votes, then the candidate with fewest first-place votes is eliminated.
- ⋆ For each voter whose top choice has been eliminated, their vote is re-allocated to their next choice.
- ⋆ If no candidate wins a majority when the re-allocated votes are included, then the candidate with fewest first-place votes is eliminated.
- ⋆ For each voter whose top choice has been eliminated, their vote is re-allocated to their next choice.
- ⋆ Eventually, there is a majority winner, who wins the election.

Maine's 2nd Congressional District 2018:
Bruce Poliquin v Jared Golden v Tiffany Bond v Will Hoar

round 1	1st	percent
Poliquin	133,954	46.4%
Golden	131,781	45.6%
Bond	16,408	5.7%
Hoar	6,778	2.3%

2nd choice of Hoar voters		
Poliquin	863	12.6%
Golden	1,172	17.3%
Bond	2,534	37.4%
—	2,209	32.6%

round 2	1st/2nd	percent
Poliquin	134,817	47.0%
Golden	132,953	46.4%
Bond	18,942	6.6%

next pick, Bond/Hoar voters		
Poliquin	3,593	14.7%
Golden	8,816	36.1%
—	12,001	49.1%

round 3	1st/2nd/3rd	percent	
Poliquin	138,410	49.4%	
Golden	141,769	50.6%	←winner

⋆ possible GPA method result: Golden > Bond> Hoar > Poliquin

1860 Presidential Election:
Abraham Lincoln v Stephen Douglas v John Breckenridge v John Bell

	1st	2nd*	3rd*	4th*
Lincoln	40%	14%	16%	30%
Douglas	29%	22%	48%	1%
Breckenridge	18%	18%	2%	61%
Bell	13%	46%	34%	8%

*William Riker *Liberalism against popularism*

- ⋆ Lincoln wins plurality*: Lincoln > Douglas > Breckenridge > Bell
  - *Electoral College: Lincoln (180) > Breckenridge (72) > Bell (39) > Douglas (12)
- ⋆ Bell wins vote-for-two: Bell > Lincoln > Douglas > Breckenridge
- ⋆ Douglas wins anti-plurality: Douglas > Bell > Lincoln > Breckenridge
- ⋆ Douglas wins GPA: Douglas > Bell > Lincoln > Breckenridge
- ⋆ Douglas wins ranked-choice: Douglas > Lincoln > Breckenridge > Bell
- ⋆ Any of Bell, Douglas, or Lincoln could have won with approval voting.

What happens when a candidate drops out?

three voters prefer	Amy	>	Cory	>	Liz	> Beto
six voters prefer	Amy	>	Liz	>	Cory	> Beto
three voters prefer	Beto	>	Cory	>	Liz	> Amy
five voters prefer	Beto	>	Liz	>	Cory	> Amy
two voters prefer	Cory	>	Beto	>	Liz	> Amy
five voters prefer	Cory	>	Liz	>	Beto	> Amy
two voters prefer	Liz	>	Beto	>	Cory	> Amy
four voters prefer	Liz	>	Cory	>	Beto	> Amy

- ⋆ Amy wins plurality: Amy (9) > Beto (8) > Cory (7) > Liz (6)
- ⋆ If Liz drops out: Cory (11) > Beto (10) > Amy (9)
- ⋆ If Cory drops out: Liz (11) > Beto (10) > Amy (9)
- ⋆ If Beto drops out: Liz (11) > Cory (10) > Amy (9)
- ⋆ If Amy drops out: Liz (12) > Cory (10) > Beto (8)
- ⋆ Liz wins GPA: Liz (58) > Cory (54) > Beto (41) > Amy (27)
- ⋆ Cory wins ranked-choice: Cory (20=7+4+9) > Beto (10=8+2+0)

Which voting system is best?

Desired properties:

- ⋆ neutral between the candidates: no candidate is more likely to win
- ⋆ anonymous among the voters: all votes count equally

  (NOT satisfied by the Electoral College: outcome may change if Mary from Maryland trades votes with Florrie from Florida.)

- ⋆ unrestricted: voters may vote however they want
- ⋆ unanimous: if everyone agrees, consensus decides the outcome
- ⋆ non-manipulable: a voter can't move Amy above Beto by lying about how they feel about Cory

Non-desired properties:

- ⋆ dictatorship: Beyoncé gets to pick the winner

Arrow's Theorem/Gibbard–Satterthwaite Theorem:

In an election with more than two candidates, any voting system that is unrestricted, unanimous, and non-manipulable must be a dictatorship!

Our plurality system is

- ⋆ unrestricted: voters may vote however they want
- ⋆ unanimous: if everyone agrees, that decides the outcome
- ⋆ not a dictatorship: no single voter gets to decide the outcome

but it is highly manipulable:

2000 Florida vote totals

George W. Bush	2,912,790	48.847%
Al Gore	2,912,253	48.838%
Ralph Nader	97,488	1.635%
Pat Buchanan	17,484	0.293%

If Nader voters had lied about their first choice,
they could have changed the result from Bush > Gore to Gore > Bush.

If no voting system is perfect how do we evaluate different methods?

Saari's test for voting systems:

- ⋆ voters whose preferences "cancel each other out" should yield ties

Votes that should result in a tied election:

- ⋆ one vote each for every possible ballot: if $n!$ voters vote for n candidates in each of the $n!$ possible orders
- ⋆ a preference cycle: if

one voter prefers	Amy	>	Beto	>	Cory
one voter prefers	Beto	>	Cory	>	Amy
one voter prefers	Cory	>	Amy	>	Beto

- ⋆ an opposing pair: if you and I prefer

cookie	>	vanilla	>	strawberry	>	chocolate	>	mint
mint	>	chocolate	>	strawberry	>	vanilla	>	cookie

With the GPA method, each of these elections results in a tie — but none of the other methods award ties to opposing pairs of votes.

Conclusions:

- ⋆ No voting system is perfect …
- ⋆ but many voting systems are better than the current plurality method.
- ⋆ No voting system will always deliver the result that you want: with nearly any other method Lincoln would have lost in 1860.
- ⋆ But it wouldn't be fair if it did: after all, we don't want to live in a dictatorship!
- ⋆ Mathematical reasoning can be used to assess and compare voting systems, and explain the paradoxical outcomes that might occur.

Election reform matters at all levels:
if you think it is hopeless to reform the Electoral College,
perhaps you can amend your local election system
or improve the voting procedure for a volunteer organization?

The mathematics of social choice includes a much broader list of topics:

- ⋆ council elections: which voting systems should be used to fill multiple vacant sets?
- ⋆ weighted voting systems: used for shareholder voting, the UN security council, and the Electoral College*
- ⋆ strategy-proof voting: what methods might encourage voters to vote their true preferences?
- ⋆ referrenda: what's the best way to structure ballot questions when voters' opinions about one proposition might depend on the outcome of another?
- ⋆ apportionment: by 2010 census, Nebraska should be awarded 2.57 of the 435 seats in the house — so how many seats is that?
- ⋆ gerrymandering: how can we evaluate whether congressional district lines have been drawn fairly?

Thank you!