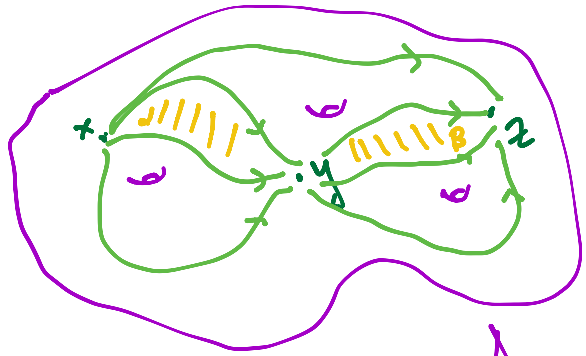
CONTRACTIBILITY as UNIQUENESS



$= A$

The standard technique used to distinguish your favorite space A from other spaces is to compute an algebraic invariant of the space.

The "algebra of paths" in a space is described in increasing precision by

- the fundamental group $\pi_1(A,x)$ of loops in A based at x up to homotopy
- the fundamental groupoid $\Pi_1 A$ of paths in A up to homotopy
- the fundamental ∞-groupoid $\Pi_\infty A$ of paths in A

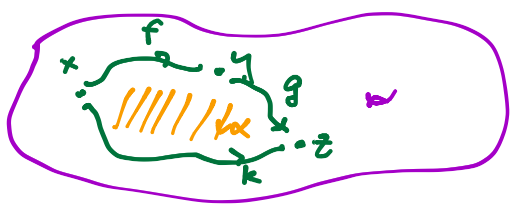
$\Pi_\infty A$ has

- points of A as objects
- paths in A as 1-dimensional arrows
- homotopies between paths in A as 2-dimensional arrows
- homotopies between homotopies between paths in A as 3-dimensional arrows, and so on...

Q: How do we define the composite of two paths? A: We don't!

Instead of a composition operation, composites of paths are witnessed by homotopies:

α is a witness that k is a composite $g \circ f$



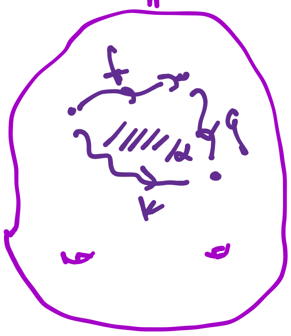
Q: How unique is this? Partial A: Unique enough for associativity:

Given composable paths f, g, h and specified homotopies witnessing composition relations, these homotopies compose.

More precisely, a 3-arrow expresses a coherence between compositions witnessed by 2-arrows.

THEOREM: The space of composites of two paths f and g in A is contractible.

proof: The first step is to define the space of composites of paths f and g in A:



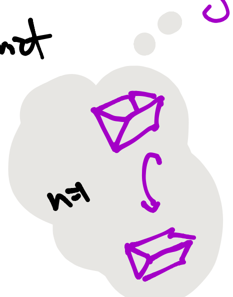
$$\begin{tikzcd} \mathrm{Comp}(f,g) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A^{\Delta} \arrow[d, "\text{restrict}"] \\ * \arrow[r, "f\wedge g"'] & A^{\Lambda} \end{tikzcd}$$

A space is contractible just when any sphere S^{n-1} can be filled in to a disk D^n $\forall n \geq 1$, so we need to define the blue map, for which it suffices to define the green map.

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d, hook] & \mathrm{Comp}(f,g) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A^{\Delta} \arrow[d, "\text{restrict}"] \\ D^n \arrow[r] \arrow[ur, dashed] \arrow[urr, dashed] & * \arrow[r, "f\wedge g"'] & A^{\Lambda} \end{tikzcd}$$

This diagram transposes to:

$$\begin{tikzcd} S^{n-1}\times\Delta \cup_{S^{n-1}\times\Lambda} D^n\times\Lambda \arrow[r] \arrow[d, hook] & A \\ D^n\times\Delta \arrow[ur, dashed] & \end{tikzcd}$$



∎

The extension exists since the inclusion admits a continuous (deformation) retraction. ∎

SUMMARY In a group(oid) any composable pair of arrows has a unique composite.
In an ∞-group(oid) any composable pair of arrows has a contractible space of composites.

The ANALOGY ordinary mathematics :: higher mathematics can be made even tighter.

uniqueness :: contractibility

To say a set C has a unique element means $\exists x \in C, \forall y \in C, x = y$

Here "$x=y$" is a predicate — a mathematical statement that is either true or false that depends on two free variables $x, y \in C$.

In proof relevant mathematics, we instead interpret "$x=y$" as the set of all proofs that x equals y (which is empty if x and y are not equal).

Then we can form the set $\sum_{x \in C} \prod_{y \in C} x=y$ inspired by a notational analogy with the sentence $\exists x \in C, \forall y \in C, x=y$.

In proof relevant mathematics, $\sum_{x \in C} \prod_{y \in C} x=y$ is also a set of proofs, but proofs of what?

An element of this set is a choice of some element $c \in C$ together with a proof, for all $z \in C$, that c equals z. In other words $\sum_{x \in C} \prod_{y \in C} x=y$ is the set of proofs that C contains a unique element (which is empty if this isn't true).

Similarly we will see that if C is a space, then $\sum_{x \in C} \prod_{y \in C} x=y$ can be interpreted as a code that defines another space. Once again we can interpret this as a space of proofs.... but proofs of what?

To explain this requires a digression to explain the analogy

logic	$\exists$	$\forall$
set theory	Σ	Π

DIGRESSION A set function $S \xrightarrow{f} T$ induces order-preserving functions between their powersets

$$\begin{tikzcd} P(S) \arrow[r, bend left, "\exists_f"] \arrow[r, bend right, "\forall_f"'] & P(T) \arrow[l, "\Delta_f"'] \arrow[l, phantom, shift right=3, "\perp"] \arrow[l, phantom, shift left=3, "\perp"] \end{tikzcd}$$

Δ_f is the inverse image: $B \subset T \mapsto \{s \in S \mid f(s) \in B\} \subset S$

$\exists_f$ is the direct image: $A \subset S \mapsto \{t \in T \mid \exists s \in S, f(s) = t \wedge s \in A\} \subset T$

$\forall_f$ is the pushforward: $A \subset S \mapsto \{t \in T \mid \forall s \in S, f(s) = t \Rightarrow s \in A\} \subset T$

For the unique function $S \xrightarrow{!} *$ these reduce to

$$\begin{tikzcd} P(S) \arrow[r, bend left, "\exists"] \arrow[r, bend right, "\forall"'] & P(*) = \{*, \phi\} \arrow[l, "\Delta"'] \arrow[l, phantom, shift right=3, "\perp"] \arrow[l, phantom, shift left=3, "\perp"] \end{tikzcd}$$

"Quantifiers as adjoints"

The powerset $P(S)$ can be identified with the set of predicates $p(s)$ with one free variable $s \in S$ — the corresponding subset is $\{s \in S \mid p(s) \text{ is true}\}$.

If we interpret the two elements of $P(*)$ by declaring $*$ means "true" and ϕ means "false"

then $\exists$ is the function that sends a predicate $p(s)$ to the sentence $\exists s \in S, p(s)$

while $\forall$ is the function that sends a predicate $p(s)$ to the sentence $\forall s \in S, p(s)$.

In proof relevant mathematics, it's better to replace the poset $P(S)$ by the category Set/S of S-indexed sets. An object in Set/S is a family of sets $\{P(s)\}_{s \in S}$ where $P(s)$ can be thought of as the set of proofs of some predicate $p(s)$ on $s \in S$.

For any function $S \xrightarrow{f} T$ there are functors

$$\begin{tikzcd} Set/S \arrow[r, bend left, "\Sigma_f"] \arrow[r, bend right, "\Pi_f"'] & Set/T \arrow[l, "\Delta_f"'] \arrow[l, phantom, shift right=3, "\perp"] \arrow[l, phantom, shift left=3, "\perp"] \end{tikzcd}$$

Δ_f is substitution: $\{Q(t)\}_{t \in T} \mapsto \{Q(f(s))\}_{s \in S}$

Σ_f is sum: $\{P(s)\}_{s \in S} \mapsto \{\sum_{s \in f^{-1}(t)} P(s)\}_{t \in T}$

Π_f is product: $\{P(s)\}_{s \in S} \mapsto \{\prod_{s \in f^{-1}(t)} P(s)\}_{t \in T}$

This gives a more formal way to understand the set $\sum_{x \in C} \prod_{y \in C} x = y$

Recall "$x=y$" is the set of proofs that x equals y where $x,y \in C$.

This should be thought of as an indexed set $\{x=y\}_{x,y\in C} \in \mathrm{Set}_{/C\times C}$.

The product along the projection function $\begin{array}{c} C\times C \xrightarrow{\pi_1} C \\ (x,y) \longmapsto x \end{array}$ gives $\left\{\prod_{y\in C} x=y\right\}_{x\in C} \in \mathrm{Set}_{/C}$.

Then the sum along $C \xrightarrow{!} *$ gives $\sum_{x\in C}\prod_{y\in C} x=y \in \mathrm{Set}_{/*} = \mathrm{Set}$.

EXTENDED ANALOGY

In logic, $\exists$ + $\forall$ are constructions on predicates

$$\begin{tikzcd} P(S) \arrow[r, bend left, "\exists_f"] \arrow[r, bend right, "\forall_f"'] & P(T) \arrow[l, "\Delta_f"] \end{tikzcd}$$

In set theory, Σ + Π are constructions on indexed sets of proofs

$$\begin{tikzcd} \mathrm{Set}_{/S} \arrow[r, bend left, "\Sigma_f"] \arrow[r, bend right, "\Pi_f"'] & \mathrm{Set}_{/T} \arrow[l, "\Delta_f"] \end{tikzcd}$$

Now replace Set by a suitably nice category of topological spaces and continuous functions.

$$\begin{tikzcd} \mathrm{Space}_{/S} \arrow[r, bend left, "\Sigma_f"] \arrow[r, bend right, "\Pi_f"'] & \mathrm{Space}_{/T} \arrow[l, "\Delta_f"] \end{tikzcd}$$

For a space C this constructs a new space $\sum_{x\in C}\prod_{y\in C} x=y$,

where an important new idea tells us to interpret "$x=y$" as the space of paths in C from the point x to the point y.

Once more, $\sum_{x\in C}\prod_{y\in C} x=y$ is a space of proofs.... but proofs of what?

Q: What is a point in the space $\sum_{x\in C} \prod_{y\in C} x=y$?

A point in $\sum_{x\in C} \prod_{y\in C} x=y$ is given by the choice of a basepoint $c\in C$ together with a point in the space $\prod_{y\in C} c=y$.

This latter point in $\prod_{y\in C} c=y$ is given by a continuous function γ from $z\in C$ to the space of paths in C from c to z.

Together this data defines a basepoint c and a contracting homotopy γ.

SUMMARY $\sum_{x\in C} \prod_{y\in C} x=y$ is the space of proofs that C is contractible!

This gives a glimpse of the meaning of uniqueness in a new proposed foundation system for mathematics called HOMOTOPY TYPE THEORY.