THURSDAY SEMINAR: HIGHER CATEGORY THEORY

NOTES BY EMILY RIEHL

ABSTRACT. The goal is to go through the Barwick–Schommer-Pries paper.

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1. INTRODUCTION (MIKE HOPKINS)

Today I want to talk about how the theory of $(\infty, 1)$-categories came to our attention. ("Wait a minute. I have to pull up my lecture preparation simulator.") It starts with a question of Quillen, in Quillen's theory of homotopical algebra.

Quillen noticed that if C is a model category you can define, for $X, Y \in C$, a set $\text{ho}C(X, Y)$. This is actually a lot like classical homotopy theory. Can form

$$\begin{tikzcd} X \arrow[r] \arrow[d] & CX \sim * \arrow[d] \\ * \arrow[r] & \Sigma X \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Just like in the classical case you have to assume these maps are pointed. So then $\text{ho}C(\Sigma X, Y)$ is a group and you can iterate and $\text{ho}C(\Sigma^2 X, Y)$ is an abelian group.

Q. Can one associate a homotopy type $\underline{\text{ho}C}(X, Y) \in \text{ho}\mathbf{Top}$ so that $\pi_0 \underline{\text{ho}C}(X, Y) = \text{ho}C(X, Y)$ and so that $\pi_k \underline{\text{ho}C}(X, Y) = \text{ho}C(\Sigma^k X, Y)$.

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The answer to this is yes (due to Dan Kan and Bill Dwyer around 1982) is yes, and this seems rather remarkable. Somehow it means that as soon as you allow a homotopy into the game you get all the higher dimensional simplices.

If the category is the category of chain complexes, these groups are Ext groups so this unifies homotopy theory with homological algebra, which was part of the design.

Dwyer-Kan showed that you can associate to $\mathcal{C}$ a category $\underline{\mathcal{C}}$ which is enriched over simplicial sets with the property that $\underline{\mathcal{C}}(X, Y)$ has the correct homotopy type. They actually had many constructions for this, some of which worked better than others for different purposes. Some were only weakly enriched. The existence of multiple constructions precipitated a need for comparison, to show that they were the "same." This is where the story for $(\infty, 1)$-categories really begins.

Specifically, Dwyer and Kan introduced a notion of weak equivalence for simplicial categories (categories enriched over simplicial sets) which are now called **DK equivalences** (the renaming maybe due to Julie Bergner?). Suppose $\underline{\mathcal{C}}$ is a category enriched over simplicial sets. Given $a, b \in \underline{\mathcal{C}}$ we have a simplicial set $\underline{\mathcal{C}}(a, b)$. Then $\pi_0\underline{\mathcal{C}}(a, b)$ is a set. Let's call the category with these hom-sets $\pi_{\leq 1}\mathcal{C}$. It's objects are the objects of $\underline{\mathcal{C}}$ and its hom-sets are $\pi_{\leq 1}\underline{\mathcal{C}}(a, b) = \pi_0\underline{\mathcal{C}}(a, b)$.[1] By analogy with homotopical algebra, they called this the homotopy category of $\underline{\mathcal{C}}$, perhaps denoted $\mathrm{ho}\underline{\mathcal{C}}$.

Definition 1.1. A **DK-equivalence** is a simplicial functor $F : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ so that

- $\mathrm{ho}F : \mathrm{ho}\underline{\mathcal{C}} \to \mathrm{ho}\underline{\mathcal{D}}$ is an equivalence of categories
- for all $a, b \in \underline{\mathcal{C}}$, the map $\underline{\mathcal{C}}(a, b) \to \underline{\mathcal{D}}(Fa, Fb)$ is a weak homotopy equivalence.

We also have an enriched homotopy category $\underline{\mathrm{ho}\mathcal{C}}$ which is enriched over $\mathrm{ho}\mathbf{sSet}$. The objects are again the objects of $\underline{\mathcal{C}}$ and $\underline{\mathrm{ho}\mathcal{C}}(a, b) = \mathcal{C}(a, b) \in \mathrm{ho}\mathbf{sSet}$, the homotopy type of the hom-space. This is a category intrinsically associated to the weak equivalence class of the simplicial category $\underline{\mathcal{C}}$.

Q. If two simplicial categories $\underline{\mathcal{C}}$ and $\underline{\mathcal{D}}$ have equivalent enriched homotopy categories are they DK-equivalent? Certainly if there is a map but the consensus is that the answer is no, maps can be obstructed.

Even in the 1980s people were already (in secret) talking about the "homotopy theory of homotopy theories" but Mike's impression is that this was regarded as a somewhat fantastical idea. Phil says this was called the "mother of all homotopy theories" but Dan didn't expect that joke would last.

Lifting problem. Kan, Dwyer, and Jeff Smith in the 1980s. Starting with an ordinary category $\mathcal{I}$ and a Quillen model category $\mathcal{C}$. Given a diagram

$$\begin{tikzcd} & \mathcal{C} \arrow[d] \\ \mathcal{I} \arrow[r] \arrow[ur, dashed] & \mathrm{ho}\mathcal{C} \end{tikzcd}$$

does there exist a lift and if so how many? What are the obstructions?

There were some antecedents to this: e.g., G a group and $\mathcal{I}$ the encoding as a one-object category.

There are some very simple examples where there are more than one lift. For example $\mathbb{Z}/2$ acting on S^1 (e.g. when you're driving). You want it to act trivially up to homotopy.

[1]This is exactly the underlying category of the enriched category $\underline{\mathcal{C}}$.

You can lift it to the trivial action. Or you can lift it to the antipodal action. And these can't be the same because the one has only fixed points and the other none.

This is very representative of Dan's work. The question is set up in such an organized way that the answer feels like the obvious one, but there's really some depth to this.

They set up a moduli space of lifts, i.e., a simplicial set whose 0-simplices are literally lifts $\mathcal{I} \to C$ that either exactly lift the original functor (or up to a specified isomorphism, but let's assume the strict thing). Calling the functor $\mathcal{F}\colon \mathcal{I} \to \mathrm{ho}C$ an n-simplex is then a sequence of lifts

$$\mathcal{F}_0 \xrightarrow{\sim} \mathcal{F}_1 \xrightarrow{\sim} \cdots \xrightarrow{\sim} \mathcal{F}_n$$

all covering the identity natural transformation of $\mathcal{F}$.

They also set up an obstruction theory. The first approach was to replace the indexing category by a simplicial category $\tilde{\mathcal{I}}_\bullet \to \mathcal{I}$ of a very special kind. Recall a simplicial category $\tilde{I}_\bullet$ can be encoded as an identity-on-objects simplicial object in **Cat**. Here, for $\tilde{\mathcal{I}}_\bullet$ each $\tilde{I}_n$ is free on some set of morphisms and furthermore the degenerate images of the generating morphisms for $\tilde{I}_k$ are among the generators for $\tilde{I}_n$ for all $n > k$.[2] The final property is that for each $a, b \in \mathcal{I}$ the map $\tilde{I}_\bullet(a, b) \to \mathcal{I}(a, b)$ is a weak equivalence (where the codomain is a discrete simplicial set).[3]

The moduli space of lifts

$$\begin{tikzcd} & C \arrow[d] \\ \tilde{I}_n \arrow[r] \arrow[ur, dashed] & \mathrm{ho}C \end{tikzcd}$$

is easy to describe because the category $\tilde{I}_n$ is free on a reflexive directed graph: You just have to lift the generating arrows.

This leads to an obstruction theory, i.e., a spectral sequence for computing the homotopy group for the moduli space of lifts, which let's call $\mathrm{lifts}_{\mathcal{F}}$. The E_2-term can be identified in Quillen-style terms.

Aside. This isn't a spectral sequence of abelian groups or anything. It can converge to the empty-set. ("It's one of these really super duper spectral sequences.") Simplicial objects in things satisfying any algebraic identities form a model category ("Theorem SA"). Categories are an example of this. There's a homology that Quillen associates to this (some derived functor of abelianization). So there's some abstracted version of cohomology.

What's interesting is the original problem

$$\begin{tikzcd} & C \arrow[d] \\ \mathcal{I} \arrow[r] \arrow[ur, dashed] & \mathrm{ho}C \end{tikzcd}$$

doesn't really have any topology involved but it naturally leads into the structures that Dwyer-Kan were thinking about. This method requires the use of simplicial categories and DK-equivalences. It is some kind of Atiyah-Hirzebruch spectral sequence for computing maps in some kind of homotopy theory of simplicial categories and DK-equivalences.

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[2] This is exactly to say that $\tilde{I}$ is cofibrant in the Bergner model structure.

[3] This, plus the fact that these things will have the same objects, is exactly to say that $\tilde{\mathcal{I}} \to \mathcal{I}$, the latter thing again the discrete simplicial category, is a DK-equivalence.

Model category of simplicial categories. Now in the 2000s, Julia Bergner shows that categories enriched over simplicial sets and DK-equivalences form a model category [Ber07a]. She advocated for this as a "homotopy theory of homotopy theories." This was asserted by Dwyer and Kan in the 1980s but they got some of the details wrong (wrong generating acyclic cofibrations?). This allows us to approach some of these problems a bit more systematically.

The Dwyer-Kan obstruction theory was a way to formulate and approach these lifting problems. But do all of the lifting problems determine the homotopy theory? By an observation of Bob Thomason: yes it does! Suppose I have two model categories $\mathcal{C}$ and $\mathcal{D}$and suppose $\text{ho}\mathcal{C} = \text{ho}\mathcal{D}$ and suppose the moduli spaces of all the lifts as the same. Solving the lifting problem

$$\begin{tikzcd} \mathcal{C} \arrow[r, dashed] \arrow[d] & \mathcal{D} \arrow[d] \\ \text{ho}\mathcal{C} \arrow[r, "="] & \text{ho}\mathcal{D} \end{tikzcd}$$

gives the map that is the missing piece.

Other approaches: We can look at strings of composable arrows in $\mathcal{I}$ and study lifts

$$\begin{tikzcd} & & \mathcal{C} \arrow[d] \\ \Delta[n] \arrow[r, "x"'] \arrow[urr, dashed] & \mathcal{I} \arrow[r, "\mathcal{F}"'] & \text{ho}\mathcal{C} \end{tikzcd}$$

Another observation of Dwyer-Kan is that

$$\text{lift}_{\mathcal{F}} = \operatorname{holim} \text{lift}_{\mathcal{F}\circ x}.$$

This motivated work of Alex Heller. Another approach is due to Charles Rezk [Rez01].

Complete Segal spaces. Associate to $\mathcal{C}$ a sequence of simplicial sets $X_0, X_1, \ldots$ where X_n is the classifying space of the category (i.e., is the category, i.e., is the nerve of the category) of all functors $\Delta[n] \to \mathcal{C}$ and whose maps are natural weak equivalences. So a j-simplex in X_n is a map from $\Delta[n] \times \Delta[j]$ so that the components in the j direction are weak equivalences.

X_n is the Dwyer-Kan moduli space of sequences of n-tuples of composable arrows. So X_0 is the moduli space of objects, X_1 is the moduli space of maps, etc. This $X_\bullet$ is a simplicial space and we have the Segal maps

$$X_n \to X_1 \times_{X_0} \cdots \times_{X_0} X_1,$$

Clark: implicitly we're doing some sort of Reedy fibrant replacement on this simplicial object. Then these maps are acyclic fibrations but already it was a weak equivalence.

Given an arbitrary simplicial space $X_\bullet$ we could take as an axiom that these maps are weak equivalences. There is one more axiom. There's a way of expressing in terms of arrows when a map is a weak equivalence (there exists an inverse and homotopies). Charles' second condition on these simplicial spaces (which should also be Reedy fibrant) is that the map $X_0 \to X_1^{\text{equiv}}$ is a weak equivalence. He calls these **complete Segal spaces**. He produced a model structure on simplicial spaces with these fibrant objects and proposes this as a model for the homotopy theory of homotopy theories.

Advantages of Rezk's theory:

- complete Segal spaces are often what naturally arises in examples
- it lends itself to computation, especially when everything is symmetric monoidal

This is a very nice theory growing out of another approach to the lifting problem.

Both Charles' and Julie's theories are really cornerstones of the homotopy theory of homotopy theories but the fact that they were both published in the *Transactions of the AMS* illustrates how under appreciated they were in their day. People really computed with these (e.g. Hopkins-Miller theories, the theory of p-compact groups). These kind of obstruction theories were used in a really practical, computational way.

There's a third point of view on this that is really striking and I belong to a community that as far as that viewpoint is concerned was really asleep at the wheel. This the theory of quasi-categories. (Independently: Hirschowitz-Simpson and Tamsemani developed a notion of Segal categories. These also have accompanying higher categorical versions.)

Quasi-categories. These were introduced by Boardman-Vogt in *Homotopy invariant algebraic structures* in the 1970s? They gave definitions of things like the space of A_∞-structures on X, or the space of E_∞-structures, or algebras over any operad, or things even more general. They thought more in terms of the algebraic theories picture than the operads picture which makes it difficult to think about what they mean by a map. They define an A_∞-structure on a map $f\colon X \to Y$ to be an A_∞-structure on the mapping cylinder $\mathrm{cyl}(f)$ that extends the A_∞-structure on the ends. But now there's a new problem: you can't compose maps. Given $X \xrightarrow{f} Y \xrightarrow{g} Z$ an A_∞-structure on g and f don't give an A_∞-structure on gf just because mapping cylinders don't compose strictly.

"Haynes and I actually ran into this in the early 90s, and I thought this was a defect in the theory." The theory was set up so that you didn't get a category. What Boardman-Vogt did instead was introduce a **quasi-category** (which they called a "weak Kan complex"; "quasi-category" is due to Joyal).

Definition 1.2. A **quasi-category** is a simplicial set X so that

$$\begin{tikzcd} V_k[n] \arrow[r] \arrow[d] & X \\ \Delta[n] \arrow[ur, dashed] & \end{tikzcd}$$

for all $n \geq 2$, $0 < k < n$, where $V_k[n] \subset \Delta[n]$ is the union of all codimension-1 faces containing the k-th vertex.

Example 1.3. In a category you can fill a $V_1[2]$ but not necessarily a $V_0[2]$ or a $V_2[2]$. Categories are quasi-categories. A category is a groupoid if and only if you can fill all the horns in its nerve.

$$\begin{tikzcd}[column sep=small] & \bullet \arrow[dr, "g"] & & & & \bullet \arrow[dr, "g"] & \\ \bullet \arrow[ur, "f"] & & \bullet & \rightsquigarrow & \bullet \arrow[ur, "f"] \arrow[rr, "gf"'] & & \bullet \end{tikzcd}$$

Joyal undertook to do all of category theory in quasi-categories (limits, colimits, etc). He introduced a model structure on simplicial sets whose fibrant objects are the quasi-categories. He wrote several hundred pages of this. Around this time Jacob Lurie came on the seen and used quasi-categories as a foundation for higher topos theory and independently developed all the basic category theory. He advocated calling these $(\infty, 1)$-categories.

There are model structures and Quillen equivalences comparing the model categories:

$$\begin{tikzcd}
\text{simplicial categories} \arrow[rr, shift left] \arrow[dr, shift left] & & \text{Rezk spaces} \arrow[ll, shift left] \arrow[dl, shift left] \\
& \text{quasi-categories} \arrow[ul, shift left] \arrow[ur, shift left] &
\end{tikzcd}$$

For Rezk spaces and quasi-categories there's a nice paper by Joyal-Tierney. A summary is in Bergner's [Ber07b]. Because these homotopy theories are equivalent, any structure that I'm looking at from any point of view, I should be able to extract from any other point of view.

Note from X a quasi-category there is an associated homotopy category $\mathrm{ho}X$ and also a category $\underline{\mathrm{ho}}X$ enriched over the homotopy category of spaces $\underline{\mathrm{ho}}(\mathbf{sSet})$. The objects are the vertices X_0. The hom-set $\mathrm{ho}X(a, b)$ is defined to be the set of 1-simplices $a \xrightarrow{f} b$ subject to the equivalence relation $f \sim f'$ whenever there exists a 2-simplex with 1st face f, 2nd face f', and 0th face degenerate at b. When X is a Kan complex, this recovers the Dwyer-Kan definition of the fundamental groupoid of X.

Kan complexes are special kinds of quasi-categories. There is a model structure (Quillen's) in which these are exactly the fibrant objects.

Theorem 1.4 (Joyal)**.** *A quasi-category is a Kan complex if and only if* $\mathrm{ho}X$ *is a groupoid.*

Because this was the fundamental group of spaces it's more natural to denote this $\mathrm{ho}X$ by $\pi_{\leq 1}X$ and call it the **fundamental category** of the quasi-category. As remarked in *Higher Topos Theory*, this makes it plain that spaces (Kan complexes) are ∞-groupoids sitting in $(\infty, 1)$-categories.

There are a number of explicit (combinatorial) models for the homotopy type $\underline{\mathrm{ho}}X(a, b)$. For instance, you could take $\underline{\mathrm{ho}}X(a, b)_n$ to be $(n + 1)$-simplices in X whose initial vertex is a and whose 0th face is degenerate at b. I gave a course on this stuff a few years ago that discussed a lot of obstruction theory.

These quasi-categories are objects that we can think of like a category and like a space. ("It's like it has the head of a space and the body of the category. I never get a laugh out of this joke. It's like a liger.")

We're calling this a homotopy theory of homotopy theories but that's just a name. What is this the homotopy theory of? It's not true that every quasi-category comes from a model category. People say this is the homotopy theory of $(\infty, 1)$-categories.

There were also many models for the homotopy theory of spaces (for instance, topological spaces, which no one would ever use) and people seemed pretty happy about this.

Q. These are three examples of equivalent homotopy theories but what are these models homotopy theories of, i.e., what are $(\infty, 1)$-categories?

This question has a really beautiful formulation and answer due to Toën.

Axiomatic characterization. Toën axiomatically characterized a theory of $(\infty, 1)$-categories and showed that the moduli space of these is $B\mathbb{Z}/2$, where the $\mathbb{Z}/2$-action sends a category to its opposite.

Again there is an antecedent: Grothendieck observed that the theory of spaces was the homotopy theory freely generated by a point. But this work raised several problems:

- how to define any model of an (∞, n)-category
- how to characterize such theories

In the first half of this semester we'll spend some time on specific models of $(\infty, 1)$- and then (∞, n)-categories. Models for the latter include

- Simpson-Tamsemani Segal n-categories
- Θ_n-spaces of Joyal, Rezk or Θ_n-sets of Hahn
- n-fold complete Segal spaces of Barwick
- complicial sets of Street,[4] Verity (early model)

Barwick–Schommer-Pries generalize Toën's theorem and prove that the moduli space of (∞, n)-categories $B(\mathbb{Z}/2)^n$.

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2. Quasi-categories as $(\infty, 1)$-categories (Emily Riehl)

My goal today is less to give a comprehensive introduction to quasi-categories as a model for $(\infty, 1)$-categories[5] but rather to give one that is as close to the ground as possible. For every statement that appears below, I'll try to either explain the proof or at least give some indication of how it is proven. This strongly influences the order of the topics. Some of what will appear below is self-plagiarized from [Rie13, Part IV], written for a class I taught here last spring. Some of the rest is copied from some joint papers with Dominic Verity, which I hope will appear soon.

Here we go!

Basic notions. Suppose a simplicial set is a quasi-category unless explicitly stated otherwise. An important feature of quasi-categories that isn't true for generic simplicial sets is that for every relation in the homotopy category and any choice of representing 1-simplices, there exists a 2-simplex that witnesses the relation. More precisely:

Proposition 2.1. *Given 1-simplicies $f, g, h \in X$, $h = gf$ in $\mathrm{ho}X$ if and only if there exists a 2-simplex in X with boundary*

$$\begin{tikzcd} & \cdot \arrow[dr, "g"] \arrow[d, phantom, "="] & \\ \cdot \arrow[ur, "f"] \arrow[rr, "h"'] & {} & \cdot \end{tikzcd}$$

We have an adjunction $\mathrm{ho}\colon \mathbf{qCat} \rightleftarrows \mathbf{Cat}\colon N$ whose right adjoint, the nerve functor, is fully faithful. Sometimes it's conventional to regard categories as quasi-categories without writing the "N." In every case we know of (certainly in every example we will mention) the quasi-categorical notion, when restricted to the full subcategory of categories, will coincide exactly with the categorical notion bearing the same name. So category theory is really a subset of quasi-category theory.

Proposition 2.2. **qCat** *is cartesian closed (and admits cotensors by arbitrary simplicial sets) with the internal hom (cotensor) given by the internal hom for simplicial sets.*

There is a bit of combinatorics that goes into the proof of this, which we will address momentarily. The obvious fact is that the larger **sSet** is cartesian closed. To my mind, the reason quasi-categories are such a convenient model of $(\infty, 1)$-categories owes largely to the fact that **sSet**, as a presheaf category, is so well behaved (in particular complete and cocomplete closed symmetric monoidal). We'll see later that a number of the objects used

[4] Street had a model of (∞, ∞)-categories that he wanted to call weak ω-categories or "womcats."

[5] A (m, n)-category is a (weak) category with cells up to dimension m so that every cell above dimension n is (weakly) invertible.

to build the category theory of quasi-categories are modeled by the analogous simplicial weighted limits.

Let us think what is being asserted by this statement. From the definition, we are asked to show that for any quasi-category X and simplicial set A there exist extensions

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & X^A \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} A \times \Lambda^n_k \arrow[r] \arrow[d] & X \\ A \times \Delta^n \arrow[ur, dashed] & \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} & X^{\Delta^n} \arrow[d] \\ A \arrow[r] \arrow[ur, dashed] & X^{\Lambda^n_k} \end{tikzcd}$$

for all $n \geq 2$, $0 < k < n$.[6] The two lifting problems correspond by adjunction. Let us think what is being asserted by the latter. We are asked to choose cylinders $\Delta^m \times \Delta^n \to X$ for each m-simplex in A in a way that is compatible with the specified horn $\Delta^m \times \Lambda^n_k \to X$ and also with previously specified cylinders $\partial\Delta^m \times \Delta^n \to X$ corresponding to the boundary of the m-simplex. In other words, inductively, we must choose extensions

$$\begin{tikzcd} \partial\Delta^m \arrow[rr] \arrow[d, "i_m"'] & & X^{\Delta^n} \arrow[d, "X^{j^n_k}"] \\ \Delta^m \arrow[r] \arrow[urr, dashed] & A \arrow[r] & X^{\Lambda^n_k} \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} \partial\Delta^m \times \Delta^n \coprod_{\partial\Delta^m \times \Lambda^n_k} \Delta^m \times \Lambda^n_k \arrow[r] \arrow[d, "i_m \hat{\times} j^n_k"'] & X \\ \Delta^m \times \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

The indicated lifting problems are again transposes, on account of the **Leibniz construction** applied to the two variable adjunction between the cartesian product and internal hom.[7] Assuming the ambient categories have the necessary pullbacks and pushouts, any two-variable adjunction

$$C(a \times b, c) \cong C(a, \underline{\text{hom}}(b, c))$$

(such as a closed monoidal structure) gives rise to a two-variable adjunction

$$C^2(f \hat{\times} g, h) \cong C^2(f, \underline{\widehat{\text{hom}}}(g, h))$$

on the arrow categories. The left adjoint is the pushout product bifunctor $-\hat{\times}-$ and the right adjoint, defined dually, might be called the pullback hom (or Leibniz hom) $\underline{\widehat{\text{hom}}}(-,-)$. For example, the map $X^{\Delta^n} \to X^{\Lambda^n_k}$ is the Leibniz hom of $\Lambda^n_k \to \Delta^n$ with $X \to *$.

Such extensions always exist on account of the following result.

Proposition 2.3 (Joyal)**.** *The pushout-product of an inner anodyne map with a cofibration is inner anodyne.*

Proof. It suffices to show this is true of the $(\partial\Delta^m \to \Delta^m)\hat{\times}(\Lambda^n_k \to \Delta^n)$'s because the bifunctor $-\hat{\times}-$ preserves colimits in each variable and the inner anodyne maps, as the left class of a weak factorization system, is weakly saturated. This can be proven directly by decomposing these monomorphisms into pushouts of inner horns (see [DS11, A.1]) or via a slick, but non-constructive, argument that proves the result as stated but doesn't tell us whether the maps $(\partial\Delta^m \to \Delta^m)\hat{\times}(\Lambda^n_k \to \Delta^n)$ are *cellular* inner anodyne (relative cell complexes built from the inner horn inclusions). □

Remark. By easy formalities involving two-variable adjunctions and lifting properties there are actually three equivalent statements here, i.e., Proposition 2.3 is equivalent to either of the following two statements:

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[6]With apologies to Mike, I have to change notation. I'll write Δ^n for his $\Delta[n]$ and write Λ^n_k for his $V_k[n]$.

[7]The name, propagandized by Dominic Verity, is inspired by Leibniz' formula for the boundary of a product of polygons.

- the pullback-hom of a cofibration with an inner fibration is an inner fibration
- the pullback-hom of an inner anodyne map with an inner fibration is a trivial fibration.

In particular, the pullback-hom of $\emptyset \to A$ and $X \to *$ is $X^A \to *$, proving that X^A is a quasi-category if X is. We have another immediate corollary.

Corollary 2.4. *If X is an ∞-category, then $X^{\Delta^n} \to X^{\Lambda^n_k}$ is a trivial fibration.*

In particular, the fiber over any point is a contractible Kan complex. This says that the spaces of fillers to a given horn is a contractible Kan complex. This is the common form taken by a homotopical uniqueness statement in ∞-category theory and is what is meant by saying something is "well defined up to a contractible space of choices."

Equivalences between quasi-categories. By an observation of Joyal, the cofibrations and fibrant objects completely determine a model structure, supposing one exists. As it turns out, again by work of Joyal, the monomorphisms and quasi-categories give rise to a model structure on simplicial sets whose weak equivalences, called simply **equivalences** when between quasi-categories, are a good notion.

Theorem 2.5 (Joyal)**.** *The cofibrations and fibrant objects completely determine a model structure.*

The following argument parallels his proof of this theorem in our particular case of interest. Supposing there is such a model structure for quasi-categories, the weak equivalences must be characterized representably as maps $f: A \to B$ that induce bijections on hom-sets in the homotopy category when homming into any quasi-category X. Because all objects are cofibrant, we can characterize the hom-sets in the homotopy category of the hypothesized model structure by use of a good cylinder object.

To that end write J for the nerve of the free-standing isomorphism.[8] Observe that $J \to *$ and hence any projection $A \times J \to A$, as its pullback, is a trivial fibration. Consequently,

$$A \sqcup A \rightarrowtail A \times J \overset{\sim}{\twoheadrightarrow} A$$

defines a very good cylinder object. Using this, by a theorem of Quillen the hom-set from A to X in the homotopy category is isomorphic to the set $[A, X]_J$ defined to be the quotient of $\hom(A, X)$ by the relation generated[9] by $f \sim g$ if there exists a diagram

$$\begin{tikzcd} A \arrow[d, "j_0"'] \arrow[dr, "f"] & \\ A \times J \arrow[r] & X \\ A \arrow[u, "j_1"] \arrow[ur, "g"'] & \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} * \arrow[d, "j_0"'] \arrow[dr, "f"] & \\ J \arrow[r] & X^A \\ * \arrow[u, "j_1"] \arrow[ur, "g"'] & \end{tikzcd} \tag{2.6}$$

So we declare a map $f: A \to B$ of simplicial sets to be an **weak equivalence** if and only if it induces a bijection $[B, X]_J \to [A, X]_J$ for all quasi-categories X. We follow Lurie and call these maps **categorical equivalences**[10] or simply **equivalences** if the source and target

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[8]This is a simplicial model for $S^\infty = B(\mathbb{Z}/2, \mathbb{Z}/2, *)$, the total space of the classifying space $K(\mathbb{Z}/2, 1) = B\mathbb{Z}/2 = \mathbb{R}P^\infty$.

[9]Indeed, the "generated" here is unnecessary because X, and hence X^A, is a quasi-category; any f and g in the same equivalence class admit such diagrams, as we shall prove momentarily.

[10]Joyal calls these "weak categorical equivalences."

are quasi-categories because no ambiguity is possible in that case. A good exercise for the reader is to show that inner anodyne maps and trivial fibrations are weak equivalences using this definition.

Theorem 2.7 (Joyal)**.** *There is a left proper cofibrantly generated model structure on simplicial sets whose cofibrations are the monomorphisms and whose fibrant objects are the quasi-categories.*

Fibrations between fibrant objects, which we shall call **isofibrations** are characterized by the right lifting property against the inner horn inclusions and the map $* \to J$, which is the nerve of the functor whose right lifting property chracterizes the isofibrations in **Cat** (hence the name). Note that the trivial fibrations are the same in Joyal's and in Quillen's model structures. Some closing remarks:

- $\mathrm{ho} \dashv N$ is a Quillen adjunction with the folk model structure on **Cat**.
- As a corollary, both adjoint functors preserve equivalences. A functor between categories is an equivalence if and only if its nerve is an equivalence.
- Categorical equivalences are weak homotopy equivalences.

Quasi-categories as $(\infty, 1)$-categories. A quick inductive definition of an $(\infty, 1)$-category is that it's something (weakly) enriched over $(\infty, 0)$-categories, i.e., ∞-groupoids, i.e., homotopy types.

Aside (the homotopy category of spaces as a base for enrichment). Because I like knowing why these types of things are true, permit me a digression on why it makes sense to enrich over the homotopy category of spaces. Everyone knows that simplicial sets is a closed symmetric monoidal category and has a compatible model structure which makes it a simplicial model category. This is Quillen equivalent to a simplicial model structure on your favorite convenient category of spaces, e.g., k-spaces or compactly generated spaces. The Quillen equivalence descends to an equivalence between the homotopy categories, which we'll call the **homotopy category of spaces** and denote by $\mathcal{H}$.

Using this simplicial model structure, there is a uniform way to construct point-set level and total derived functors of left and right Quillen functors, bifunctors, etc: Just precompose with cofibrant replacement or fibrant replacement, as appropriate. The fact that the model structure is closed monoidal implies that the cartesian product and internal hom are amenable to such deformations, so have derived functors constructed in this way.

We'd like to say that the total derived functors of the closed symmetric monoidal structure on **sSet** define a closed symmetric monoidal structure on $\mathcal{H}$. To prove this we need to show that we can also derived the natural isomorphisms expressing coherence of the derived monoidal product, existence of the derived adjunction, and so forth. Now composing derived functors is somewhat non-trivial but in this case the axioms that establish that **sSet** is a monoidal model category (plus Ken Brown's lemma) say things like the hom-space from a cofibrant object to a fibrant object is again fibrant which imply that everything works out. (See [Rie13, Chapter 10] for more details.)

Furthermore, the localization functor $\mathbf{sSet} \to \mathcal{H}$ is lax monoidal which means any simplicial enrichment descends to an $\mathcal{H}$-enrichment.

Our goal is define $\underline{\mathrm{ho}}X$ as an $\mathcal{H}$-category so the underlying category — whose arrows are homotopy classes of maps $* \to \underline{\mathrm{ho}}X(a, b)$, i.e., whose hom-sets can be computed by applying π_0 to the hom-spaces — is $\mathrm{ho}X$.

The first, to my mind most obvious construction, makes use of the quasi-category X^{Δ^1} of paths in X; vertices are 1-simplices in X, and n-simplices are cylinders $\Delta^n \times \Delta^1 \to X$.

One candidate mapping space between two fixed vertices $x, y \in X$ is the pullback

$$\begin{tikzcd} \mathrm{Hom}_X(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X^{\Delta^1} \arrow[d] \\ * \arrow[r, "{(x,y)}"'] & X \times X \cong X^{\partial\Delta^1} \end{tikzcd}$$

By the combinatorics encoded above by Proposition 2.3, $\mathrm{Hom}_X(x,y)$ is a quasi-category. An n-simplex is $\mathrm{Hom}_X(x,y)$ is a map $\Delta^n \times \Delta^1 \to X$ such that the image of $\Delta^n \times \{0\}$ is degenerate at x and and the image of $\Delta^n \times \{1\}$ is degenerate at y. In particular, 1-simplices look like

(2.8)
$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, equal] \arrow[dr] \arrow[dr, phantom, "\sim", shift right=2ex] \arrow[dr, phantom, "\sim"', shift left=2ex] & y \arrow[d, equal] \\ x \arrow[r, "g"'] & y \end{tikzcd}$$

from which we see that $\pi_0\mathrm{Hom}_X(x,y)$ is the hom-set from x to y in hX.

A less symmetric but more efficient construction is also possible. Let $\mathrm{Hom}^R_X(x,y)$ be the simplicial set whose 0-simplices are 1-simplices in X from x to y, whose 1-simplices are 2-simplices of the form

$$\begin{tikzcd} & x \arrow[dr] & \\ x \arrow[ur, equal] \arrow[rr] & & y \end{tikzcd}$$

and whose n-simplices are $(n+1)$-simplices whose last vertex is y and whose $(n+1)$th face is degenerate at x. Dually, $\mathrm{Hom}^L_X(x,y)$ is the simplicial set whose n-simplices are $(n+1)$-simplices in X whose first vertex is x and whose d^0-face is degenerate at y. Once again, note that $\pi_0\mathrm{Hom}^L_X(x,y) = \pi_0\mathrm{Hom}^R_X(x,y) = hX(x,y)$.

Remark. The spaces $\mathrm{Hom}^L_X(x,y)$ and $\mathrm{Hom}^R_X(x,y)$ are dual in the sense that $\mathrm{Hom}^L_X(x,y) = (\mathrm{Hom}^R_{X^{\mathrm{op}}}(y,x))^{\mathrm{op}}$. The annoying fact, from the perspective of homotopy (co)limits, that a simplicial set is not isomorphic to its opposite, in which the conventions on ordering of vertices in a simplex are reversed, is technically convenient here.

In fact all three of these candidate hom-spaces are good models: they're all Kan complexes (the explanation for which we'll postpone for now) and they're all equivalent. To explain the equivalence, let us think geometrically about the difference.[11] Each simplicial set has the same zero simplices. An n-simplex in $\mathrm{Hom}^L_X(x,y)$ or $\mathrm{Hom}^R_X(x,y)$ is an $(n+1)$-simplex in X one of whose faces is degenerate. Thus the relevant shapes are given by the quotients

$$\begin{tikzcd} \Delta^n \arrow[r] \arrow[d, "d^0"'] & \Delta^0 \arrow[d] \\ \Delta^{n+1} \arrow[r] & C^n_L \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \qquad \begin{tikzcd} \Delta^n \arrow[r] \arrow[d, "d^{n+1}"'] & \Delta^0 \arrow[d] \\ \Delta^{n+1} \arrow[r] & C^n_R \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

This simplicial set has two vertices and has a non-degenerate k-simplex for each non-degenerate k-simplex of Δ^n whose image surjects onto Δ^1.

[11]This proof is due to Daniel Dugger and David Spivak with some modifications by Verity.

Similarly, the shape of an n-simplex in $\mathrm{Hom}_X(x,y)$ is given by

$$\begin{tikzcd} \Delta^n \times \partial\Delta^1 \arrow[r] \arrow[d] & \partial\Delta^1 \cong * \sqcup * \arrow[d] \\ \Delta^n \times \Delta^1 \arrow[r] & C^n_{\mathrm{cyl}} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

We have canonical maps

$$\begin{tikzcd} C^n_L \arrow[r] \arrow[dr] & C^n_{\mathrm{cyl}} \arrow[d] & C^n_R \arrow[l] \arrow[dl] \\ & \Delta^1 & \end{tikzcd}$$

where the horizontal maps are induced by the inclusions of $\Delta^{n+1} \rightrightarrows \Delta^n \times \Delta^1$ as the first and last shuffles respectively.[12]

These constructions define three cosimplicial objects $C^\bullet_L, C^\bullet_{\mathrm{cyl}}, C^\bullet_R$ taking values in the category of simplicial sets and maps preserving two chosen basepoints. Write $\mathbf{sSet}_{*,*}$ for this slice category $\partial\Delta^1/\mathbf{sSet}$. The simplicial set X with chosen vertices x, y becomes an object of $\mathbf{sSet}_{*,*}$. The three hom-spaces introduced above are defined from these cosimplicial objects and the hom-sets of $\mathbf{sSet}_{*,*}$ by the equalities

$$\begin{aligned} \mathrm{Hom}^L_X(x,y) &= \mathbf{sSet}_{*,*}(C^\bullet_L, X) \\ \mathrm{Hom}_X(x,y) &= \mathbf{sSet}_{*,*}(C^\bullet_{\mathrm{cyl}}, X) \\ \mathrm{Hom}^R_X(x,y) &= \mathbf{sSet}_{*,*}(C^\bullet_R, X). \end{aligned}$$

The natural maps $\mathrm{Hom}^L_X(x,y) \leftarrow \mathrm{Hom}_X(x,y) \to \mathrm{Hom}^R_X(x,y)$ come from the maps between the cosimplicial objects. We would like to show that these are equivalences. Morally, this follows because $C^\bullet_L$, $C^\bullet_{\mathrm{cyl}}$, and $C^\bullet_R$ are cofibrant resolutions of Δ^1 in the Joyal model structure. Let us give just a few more details.

Remark. The category $\mathbf{sSet}_{*,*}$, defined as a slice category, inherits a model structure from the quasi-categorical model structure on $\mathbf{sSet}$: A map of twice-based simplicial sets is a cofibration, fibration, or weak equivalence just when the underlying map of simplicial sets is one. Fibrant objects are quasi-categories with chosen basepoints. An object is cofibrant if and only if its two chosen basepoints are distinct.

Lemma 2.9. *$C^\bullet_R$, $C^\bullet_L$, $C^\bullet_{\mathrm{cyl}}$ are Reedy cofibrant.*

Proof. There's a simple criterion ("unaugmentable" in [BK72]) that detects when a cosimplicial object is Reedy cofibrant, and if you know it, it's easy to check that it's true here. □

The geometrical heart of the argument is in the proof of the following result.

Lemma 2.10. *The canonical maps $C^\bullet_L \to C^\bullet_{\mathrm{cyl}} \leftarrow C^\bullet_R$ are pointwise categorical equivalences.*

Proof. Those with patience for combinatorics can check that $C^n_L \to \Delta^1$, $C^n_{\mathrm{cyl}} \to \Delta^1$, and $C^n_R \to \Delta^1$ by showing that sections are (cellular) inner-anodyne maps [DS11]. □

[12]Recall simplices in $\Delta^n \times \Delta^m$ correspond bijectively to totally ordered collections of vertices (i,j) with $i \in [n]$ and $j \in [m]$. Simplices of maximal dimension are called **shuffles**. The first shuffle is the unique one containing the vertices $(0,0), \ldots, (n,0), \ldots (n,m)$. The last is the unique one containing the vertices $(0,1), \ldots, (0,m), \ldots, (n,m)$.

We might think about these cosimplicial spaces as "weights" whose weighted limits define our three candidate mapping spaces. To use this information to obtain our desired conclusion, the starting point is that one can define simplicial mapping spaces for $\mathbf{sSet}_{*,*}$ so that when X is a quasi-category $\underline{\hom}(-, X) \colon \mathbf{sSet}_{*,*}^{\mathrm{op}} \to \mathbf{sSet}$ is a right Quillen functor. By Ken Brown's lemma, it follows that this functor preserves equivalences between objects with distinct basepoints. The proof is completed by some Reedy category theory.

Consider a cosimplicial object $C^\bullet \colon \mathbb{\Delta} \to \mathbf{sSet}_{*,*}$. Latching and matching objects can be defined to be certain (dual) weighted colimits and limits from which it is clear that

$$M_n \underline{\hom}(C^\bullet, X) \cong \underline{\hom}(L^n C^\bullet, X).$$

If $C^\bullet$ is Reedy cofibrant, the maps $L^n C^\bullet \to C^n$ are cofibrations and hence

$$\underline{\hom}(C^\bullet, X) \to \underline{\hom}(L^n C^\bullet, X) \cong M_n \underline{\hom}(C^\bullet, X)$$

are fibrations because $\underline{\hom}(-, X)$ is right Quillen. This says that $\underline{\hom}(C^\bullet, X)$ is Reedy fibrant. Applying this result to the cosimplicial objects $C_L^\bullet, C_{\mathrm{cyl}}^\bullet, C_R^\bullet$ we see that we have pointwise weak equivalences between Reedy fibrant objects

$$\underline{\hom}(C_L^\bullet, X) \leftarrow \underline{\hom}(C_{\mathrm{cyl}}^\bullet, X) \to \underline{\hom}(C_R^\bullet, X)$$

in the category of bisimplicial sets.

Remembering only the vertices of each simplicial set in the simplicial objects — a process which might be called "taking vertices pointwise" — we are left with the diagram of simplicial sets $\mathrm{Hom}_X^L(x, y) \leftarrow \mathrm{Hom}_X(x, y) \to \mathrm{Hom}_X^R(x, y)$ that is actually of interest. The proof that these maps are weak equivalences is completed by the following lemma.

Lemma 2.11. *Suppose $f \colon X \to Y$ is a weak equivalence between Reedy fibrant bisimplicial sets. Then the associated map of simplicial sets $X_{\bullet,0} \to Y_{\bullet,0}$ obtained by taking vertices pointwise is a weak equivalence.*

Proof. By Ken Brown's lemma, it suffices to prove that if $f \colon X \to Y$ is a Reedy trivial fibration of bisimplicial sets then the associated map $X_{\bullet,0} \to Y_{\bullet,0}$ is a weak equivalence. Indeed, this map is a trivial fibration of simplicial sets. Because f is a Reedy trivial fibration, each relative matching map $X_n \to Y_n \times_{M_n Y} M_n X$ is a trivial fibration of simplicial sets, and in particular, the map on vertices $X_{n,0} \to (Y_n \times_{M_n Y} M_n X)_0 = Y_{n,0} \times_{(M_n Y)_0} (M_n X)_0$ is a surjection in $\mathbf{Set}$. But this says exactly that any lifting problem

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & X_{\bullet,0} \arrow[d] \\ \Delta^n \arrow[r] & Y_{\bullet,0} \end{tikzcd}$$

has a solution. □

Thus, we have proven:

Theorem 2.12. *The natural maps $\mathrm{Hom}_X^L(x, y) \leftarrow \mathrm{Hom}_X(x, y) \to \mathrm{Hom}^R(x, y)$ are equivalences of quasi-categories.*

Using retractions to the maps $C_L^n \to C_{\mathrm{cyl}}^n \leftarrow C_R^n$, which can be defined as quotients of the appropriate projections $\Delta^n \leftarrow \Delta^n \times \Delta^1 \to \Delta^n$, there are also equivalences $\mathrm{Hom}_X^L(x, y) \to \mathrm{Hom}_X(x, y) \leftarrow \mathrm{Hom}_X^R(x, y)$. We'll see shortly that any equivalence $X \to Y$ of quasi-categories has an inverse equivalence $Y \to X$.

Q (for the audience)**.** Reedy category theory is good for this sort of thing and for proving simplified formulas for homotopy limits and colimits. What else?

Because equivalences between quasi-categories are homotopy equivalences, the objects $\mathrm{Hom}^L_X(x,y)$, $\mathrm{Hom}_X(x,y)$, and $\mathrm{Hom}^R_X(x,y)$ define weakly equivalent simplicial sets whose set of path components is the hom-set $\mathrm{ho}X(x,y)$. We would like to conclude that the homotopy category $\mathrm{ho}X$ is thereby enriched over the homotopy category of spaces — however, there is no natural composition law definable in **sSet** using any of these mapping spaces. These considerations motivate the introduction of a fourth candidate mapping space, which is weak homotopically equivalent (but not categorically equivalent) to these models, and associates to each simplicial set a simplicially enriched category.

Homotopy coherent diagrams. The point is there is an adjunction $\mathfrak{C}\colon \mathbf{sSet} \rightleftarrows \mathbf{sCat}\colon \mathbb{N}$ between simplicial sets and simplicial categories. It is a Quillen equivalence with respect to the Joyal and Bergner model structures. In particular, if $\underline{C}$ is a locally Kan simplicial category then $\mathbb{N}\underline{C}$ is a quasi-category. This is important source of quasi-categories in practice; for instance, the quasi-category associated to a simplicial model category is defined by applying $\mathbb{N}$ to the subcategory of fibrant-cofibrant objects. On the other side, if X is a quasi-category then the hom-spaces of $\mathfrak{C}X$, while not fibrant,[13] do have the same weak homotopy type as the mapping spaces introduced above. So we can use $\mathfrak{C}X$ to define $\underline{\mathrm{ho}}X$. In particular $\mathrm{ho}X = (\pi_0)_*\mathfrak{C}X$. A consequence of this Quillen equivalence, or really rather an ingredient in the proof, is that $X \to Y$ is a categorical equivalence (of simplicial sets even) if and only if $\mathfrak{C}X \to \mathfrak{C}Y$ is a DK-equivalence.

As an expository note, Lurie's entire approach to the proof of the model structure on quasi-categories is designed to facilitate the proof that this adjunction is a Quillen equivalence, which should serve as some indication of its importance [Lur09, Chapter 2].

A lot of you know a lot about this (and some subset of you have heard me talk about this before) so I'm not going to say too much except to remind you how this adjunction is defined. The reason I want to do this is that it connects back to the story about homotopy coherence mentioned by Mike last time that motivated the development of $(\infty,1)$-category theory and quasi-categories in particular. In particular, the replacement of the indexing category $\mathcal{I}$ of a diagram by a simplicial category $\tilde{\mathcal{I}}_\bullet$ that was used to set up the obstruction theory for lifting diagrams in the homotopy category is an instance of cofibrant replacement in this model structure. Even more precisely, the map $\tilde{\mathcal{I}}_\bullet \to \mathcal{I}$ is isomorphic to the component of the counit of the adjunction $\mathfrak{C} \dashv \mathbb{N}$ at the discrete simplicial category $\mathcal{I}$.

There are two isomorphic descriptions of this cofibrant replacement. One, as I just claimed is $\mathfrak{C}N\mathcal{I}$.[14] But since we haven't defined these things yet, I'll give the other, which is the construction of Dwyer-Kan. There is a comonad F on **Cat** which replaces a category by the category freely generated by its underlying reflexive directed graph (forgetting composites but remembering identities). Note that $\mathcal{I}$ and $F\mathcal{I}$ have the same objects. Non-identity morphisms in $F\mathcal{I}$ are strings of composable non-identity morphisms. The counit $F\mathcal{I} \to \mathcal{I}$ composes the arrows in each string. The cosimplicial object in **Cat** that defines the simplicial category serving as the cofibrant replacement of $\mathcal{I}$ is the comonad resolution (augmented by this $F\mathcal{I} \to \mathcal{I}$). The n-th category is $F^{n+1}\mathcal{I}$. Its objects are the same as the objects of $\mathcal{I}$ and its morphisms are strings of composable arrows enclosed in exactly

[13] Even though the hom-spaces of $\mathfrak{C}X$ aren't fibrant, they are, in some weird sense, close. More precisely, for any simplicial set X, the hom-spaces of $\mathfrak{C}X$ are 3-coskeletal, which implies that any horn of dimension 5 or higher will have a *unique* filler. But it is easy in toy examples to find low dimensional horns that cannot be filled.

[14] The nerve and the homotopy coherent nerve coincide for discrete simplicial categories.

n pairs of parentheses (each indicating a layer of formal composition). The degeneracy maps "double up on parentheses" while the face maps remove parentheses, which should be thought of as a form of composition (because it is).

Now the adjunction $\mathfrak{C}\colon \mathbf{sSet} \rightleftarrows \mathbf{sCat}\colon \mathbb{N}$, like any adjunction so that the domain of the left adjoint is simplicial sets, is given by some "geometric realization–total singular complex"-type construction (or, if you will, "left Kan extension–nerve") with respect to some cosimplicial object $\mathbb{\Delta} \to \mathbf{sCat}$. This simplicial object is defined by taking the finite ordinal categories $[n]$ to their cofibrant replacements defined in this way. For example, let's compute the cofibrant replacement of $[2] = \mathbb{3}$, which is the category whose non-identity morphisms we might label as:

$$\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[rr, bend right, "j"'] \arrow[rrr, bend right=60, "\ell"'] & \cdot \arrow[r, "g"] \arrow[rr, bend right, "k"'] & \cdot \arrow[r, "h"] & \cdot \end{tikzcd}$$

Let us describe the hom-space from the initial object to the terminal one. The vertices of this simplicial set are the paths of edges ℓ, kf, hj, hgf. The 1-simplices are once parenthesized strings of composable morphisms which are non-degenerate when there is more than one arrow inside some pair of parentheses. There are five such with boundary 0-simplices illustrated below

(2.13)
$$\begin{tikzcd} \ell \arrow[r, "(kf)"] \arrow[d, "(hj)"'] \arrow[dr, "(hgf)" description] \arrow[dr, phantom, shift left=3ex, "((hg)(f))"] \arrow[dr, phantom, shift right=3ex, "((h)(gf))"] & kf \arrow[d, "(hg)(f)"] \\ hj \arrow[r, "(h)(gf)"'] & hgf \end{tikzcd}$$

There are only two non-degenerate 2-simplices whose boundaries are depicted above. Hence the hom-space is $\Delta^1 \times \Delta^1$.[15]

Those who are familiar with the classical literature on homotopy coherent diagrams will recognize a lot of these ideas. In the language of Cordier-Porter, Vogt, and others, a diagram of shape $\mathfrak{C}\mathbb{N}\mathcal{I}$ is exactly a **homotopy coherent diagram** of shape $\mathcal{I}$. In the context of quasi-category theory, Jacob defines a homotopy coherent diagram in a quasi-category X to be any map $N\mathcal{I} \to X$. (This makes sense geometrically if you think about the higher simplices of the nerve.) Note if X is one of these quasi-categories which arises as $\mathbb{N}\underline{C}$ for some locally Kan simplicial category $\underline{C}$ (and indeed all quasi-categories are equivalent to some such thing), then by adjunction $N\mathcal{I} \to \mathbb{N}\underline{C}$ is exactly $\mathfrak{C}\mathbb{N}\mathcal{I} \to \underline{C}$, i.e., a homotopy coherent diagram in the quasi-category is a homotopy coherent diagram in the associated simplicial category (which is another model for the $(\infty, 1)$-category).

Isomorphisms in quasi-categories. What I'm proposing here is not standard terminology but was suggested to me recently by Dominic Verity in the context of a paper we're writing. I thought I'd use it today to gauge reactions from the audience.

We say a 1-simplex in a quasi-category is an **isomorphism** if and only if it represents an isomorphism in ho$X$. By remarks made above, for any isomorphism $f\colon x \to y$ we can

[15]Those of you who have heard me talk about this sort of thing before will know that this isn't my favorite way to think about these hom-spaces: It's the "necklace" characterization of Dugger-Spivak.

choose an inverse isomorphism $g\colon y \to x$ together with 2-simplices

$$\begin{tikzcd} & y \arrow[dr, "g"] \arrow[d, phantom, "="] & \\ x \arrow[ur, "f"] \arrow[rr, equal] & {} & x \end{tikzcd} \qquad \begin{tikzcd} & x \arrow[dr, "f"] \arrow[d, phantom, "="] & \\ y \arrow[ur, "g"] \arrow[rr, equal] & {} & y \end{tikzcd}$$

A key combinatorial lemma, due to Joyal, says that quasi-categories admit "special outer horn fillers," that is, any horn $\Lambda^n_0 \to X$ can be filled provided that its initial edge is an isomorphism and dually any $\Lambda^n_n \to X$ whose final edge is an isomorphism has a filler [Joy02]. Conversely (and this part is obvious) these extension properties characterize the isomorphisms. There is also this immediate corollary:

Corollary 2.14 (Joyal)**.** *X is a Kan complex if and only if X is a quasi-category and* $\mathrm{ho}X$ *is a groupoid.*

Another corollary is that the three models for mapping spaces mentioned above are Kan complexes. The spaces $\mathrm{Hom}^L_X(x, y)$ and $\mathrm{Hom}^R_X(x, y)$ are defined as pullbacks of right fibrations, which implies that all of their edges are isomorphisms. We've shown these are equivalent to $\mathrm{Hom}_X(x, y)$ which implies that their homotopy categories are equivalent which implies that $\mathrm{hoHom}_X(x, y)$ is a groupoid which implies that $\mathrm{Hom}_X(x, y)$ is also a Kan complex.

Also:

Lemma 2.15 (Joyal)**.** *$f\colon \Delta^1 \to X$ is an isomorphism if and only if there exists an extension to $J = N(\bullet \cong \bullet)$.*

Proof. We make use of the following observation: an n-simplex in the nerve of a category is degenerate if and only if one of the edges along its spine is an identity. In particular, there are only two non-degenerate simplices in each dimension in J and furthermore, if σ is a non-degenerate n-simplex, only its 0th and nth faces are non-degenerate.

The map f lands in ιX; it therefore suffices to show that $\Delta^1 \to J$ is anodyne. In fact, we will give a cellular decomposition of this inclusion, building J by attaching a sequence of outer horns. Abusing terminology, we will call the non-degenerate 1-simplex f. The first attaching map $\Lambda^2_2 \to \Delta^1$ has 0th face f and 1st face an identity. Call the 1-simplex obtained by pushing out

$$\begin{tikzcd} \Lambda^2_2 \arrow[r] \arrow[d] & \Delta^1 \arrow[d] \\ \Delta^2 \arrow[r] & \cdot \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

g. This also defines the non-degenerate 2-simplex whose spine is fg. Next we use the Λ^3_3 horn whose boundary is depicted

$$\begin{tikzcd} & \cdot \arrow[dd, "g"] \arrow[dr, equal] & \\ \cdot \arrow[ur, "f"] \arrow[rr, "f"] \arrow[dr, equal] & & \cdot \\ & \cdot \arrow[ur, "f"'] & \end{tikzcd}$$

to obtain the non-degenerate 2-simplex with spine gf and the non-degenerate 3-simplex with spine fgf. Next attach a Λ^4_4 horn and so on. $\square$

We say that two objects in a quasi-category are **isomorphic** if and only if there is an isomorphism between them. (Exercise: check that this is an equivalence relation.) For instance, suppose $f : X \to Y$ is an equivalence between quasi-categories. In particular, it induces isomorphisms

$$[Y, X]_J \xrightarrow{f^*} [X, X]_J \qquad\qquad [Y, Y]_J \xrightarrow{f^*} [X, Y]_J.$$

Considering the first of these, we conclude that the identity on X is isomorphic in the quasi-category X^X to a vertex in the image of f. By Lemma 2.15, this isomorphism is represented by a map as displayed on the left

$$\begin{tikzcd} X \arrow[d] \arrow[dr, "1"] & & & Y \arrow[d] \arrow[dr, "1"] & \\ X \times J \arrow[r] & X & & Y \times J \arrow[r] & Y \\ X \arrow[u] \arrow[r, "f"'] & Y \arrow[u, "g"'] & & Y \arrow[u] \arrow[r, "g"'] & X \arrow[u, "f"'] \end{tikzcd} \tag{2.16}$$

Post-composing the equivalence with f we see that f and fgf are isomorphic in Y^X. From the second bijection, it follows that fg is isomorphic to the identity on Y via a map as displayed on the right above.

Some more facts whose proofs are now easy exercises:

- **qCat** has a full coreflective subcategory **Kan**. The coreflector takes a quasi-category to its maximal sub Kan complex spanning the isomorphisms.
- Any equivalence restricts to an equivalence between maximal sub Kan complexes.
- Conversely, any weak homotopy equivalence between maximal sub Kan complexes extends to a simplicial homotopy equivalence. The representing 1-simplex is an isomorphism in the hom Kan complex and hence this simplicial homotopy equivalence is a categorical equivalence (which is a priori stronger).

There are some other facts about isomorphisms that I want to mention though these are harder to prove. The proofs I know make use of the marked model structure, which again many of you know about, and in any case I don't want to get into.

Theorem 2.17 (pointwise natural isomorphisms are isomorphisms)**.** *Suppose given a natural transformation, i.e., a diagram $\Delta^1 \to X^A$. If this is a pointwise isomorphism (for each $a \in A$) then it's an isomorphism in X^A.*

This is a really awesome result, which follows essentially from the cartesian closure of the marked model structure.

Theorem 2.18 (inverting diagrams)**.** *Suppose K is any simplicial set and you have a diagram $K \to X$ in a quasi-category whose edges are taken to isomorphisms. Then this diagram admits an extension to the groupoidification $\tilde{K}$.*[16]

Any simplicial set is a colimit indexed over its category of simplicies of the Yoneda embedding. The groupoidification is formed by replacing the Yoneda embedding here by the nerves of the groupoidifications of the ordinal categories.

[16]Gijs Heuts points out (again) that there is a simpler model categorical proof of this.

Quasi-categories and Rezk spaces. Actually what I want to talk about is an analog of the Segal condition and of the completeness condition. We're going to approach this via weighted limits and now seems as good a time as any since I've just secretly brought up weighted colimits. I learned about all of this from Dominic Verity, though it's likely that related ideas have appeared elsewhere.[17]

Let $\mathcal{M}$ be a combinatorial model category (so I can perform left Bousfield localization). You might be familiar with Dugger's procedure to replace this by a Quillen equivalent simplicial model category (which is how we'd get at the quasi-category that has the same homotopy theory). But I want to do something else.

Given a diagram, for us $\mathbb{\Delta}^{\mathrm{op}} \to \mathcal{M}$, a **weighted limit** is something that represents not just cones over the diagram but cones of some arbitrary shape. This is really important for enriched category theory but actually at the moment I don't need the enriched notion of a weighted limit, just the set-based one will do. So what I mean by cones of an arbitrary shape is that at each object of the diagram I can choose how many legs of the cone point toward it and then I can specify what sort of commutativity relations are satisfied by these legs and the maps in the diagram. This is all done by means of a functor $W\colon \mathbb{\Delta}^{\mathrm{op}} \to \mathbf{Set}$ called the **weight**. The cardinality of the image of $[n]$ in the weight tells us how many legs should be above the object in the image of $[n]$ in the diagram. The maps then say which things compose with which maps in the diagram to which things.

Note of course that in this case the weight is just a simplicial set. Assuming $\mathcal{M}$ is complete, as is the case here, weighted limits always exist and can be computed as the functor cotensor product of the diagram with the weight.

Example 2.19. By the Yoneda lemma, the limit of $X\colon \mathbb{\Delta}^{\mathrm{op}} \to \mathcal{M}$ weighted by Δ^n is just the object X_n.

Example 2.20. By inspection, the limit of $X\colon \mathbb{\Delta}^{\mathrm{op}} \to \mathcal{M}$ weighted by $\partial\Delta^n$ is the n-th matching object M_nX, in other words, the object of boundary data associated to a hypothetical (but possibly non-existent) n-simplex in X.

Some general facts about weighted limits make this second example less surprising. The first observation is that weighted limits are contravariant in the weight. For instance, the matching map $X_n \to M_nX$ is the map between weighted limits induced by the canonical inclusion $\partial\Delta^n \to \Delta^n$. The second, and really the main thing, immediate from the defining universal property that I didn't state, is that weighted limits are cocontinuous in the weight. The simplicial set $\partial\Delta^n$ is built by gluing a collection of $(n-1)$-simplices together along the $(n-2)$-simplices that serve as their pairwise intersections. So the weighted limits is then the limit of the corresponding diagram of objects X_{n-1} and X_{n-2}, which is exactly the usual definition of the matching object. In practice, this means it's easy to define "made-to-order" weights whose weighted limits are whatever you want. The fact that the weight $\partial\Delta^n$ has the "shape" of the thing you're trying to describe in the weighted limit is no coinicidence.

Let me write $\lim^W X$ for these weighted limits. Other common notation (which I secretly prefer) is $\{W, F\}$.

Definition 2.21. Let $\mathcal{M}$ be a model category. Say $X \in \mathcal{M}^{\mathbb{\Delta}^{\mathrm{op}}}$ is

- **Reedy fibrant** if $\lim^{\Delta^n} X \to \lim^{\partial\Delta^n} X$ is a fibration for all n
- a **Segal space** if it is Reedy fibrant and if $\lim^{\Delta^n} X \to \lim^{\Lambda^n_k} X$ is a trivial fibration for all $0 < k < n$, or equivalently, if $\lim^{\Delta^n} X \to \lim^{\Delta^1\vee\cdots\vee\Delta^1} X$ is a trivial fibration for all n

[17]Another disclaimer: My memory of his proof is imperfect, so any errors in the following are mine.

- a **Rezk space** if it is a Segal space and if $\lim^J X \to \lim^{\Delta^0} X$ is a trivial fibration.

Notes: The first definition is isomorphic to the standard one. The second reduces to the standard one for $\mathcal{M} = \mathbf{sSet}$. Note these maps are automatically fibrations if X is Reedy fibrant because of standard lifting arguments involving adjunctions and the fact that the maps between weights are cofibrations. Here the $\Delta^1 \vee \cdots \vee \Delta^1$ is meant to be the spine of the n-simplex, built by gluing together n 1-simplices along their source and target vertices. By cocontinuity, the corresponding weighted limit of X is exactly the usual $X_1 \times_{X_0} \times \cdots \times_{X_0} X_1$.

Finally, for completeness, note by the example above that $\lim^{\Delta^0} X = X_0$. In the context of quasi-categories or Kan complexes, this $\lim^J X$ is a good candidate for the thing called X^{equiv} before; it's the object of 1-simplices that are equivalences (isomorphisms). Since we already know that this map is a fibration, by the 2-of-3 property we could deduce that it's a weak equivalence iff this is true of the monomorphism $\lim^{\Delta^0} X \to \lim^J X$, which is how the completeness condition (or univalence axiom) is usually stated.

The reason we've stated this in this form is that our goal is to prove the following theorem:

Theorem 2.22 (Verity)**.** *If $\mathcal{M}$ is combinatorial and left proper, then the left Bousfield localization of the Reedy model structure on $\mathcal{M}^{\mathbb{\Delta}^{\mathrm{op}}}$ at the pushout products of generating cofibrations in $\mathcal{M}$ with the generating trivial cofibrations in* **sSet** *gives what we might call the **model structure for Rezk objects**. These are exactly the fibrant objects. The result is a tensored, cotensored, and enriched simplicial category that is enriched as a model category over Joyal's model structure for quasi-categories.*

Let's call the axioms analogous to "SM7" with respect to the Joyal model structure "JM7," where we number them so that the only difference is between SM7(iii) and JM7(iii). For any model category the standard simplicial tensor, cotensor, and enrichment on $\mathcal{M}^{\mathbb{\Delta}^{\mathrm{op}}}$ satisfies the common 2/3rds of SM7 and JM7 [Dug01, 4.4-5]. When we localize we change the trivial cofibrations in $\mathcal{M}^{\mathbb{\Delta}^{\mathrm{op}}}$ so we have to re-prove SM7(ii), but we have the following simplification:

Lemma 2.23. *Let $\mathcal{K}$ be a tensored, cotensored, and simplicially enriched and a model category.*

(i) Given JM7(i), if cotensoring with any simplicial set preserves fibrations between fibrant objects then JM7(ii) holds.
(ii) If $\mathcal{K}$ is left proper, given JM7(i) and JM7(ii), then if for any trivial cofibration $K \to L$ in Joyal's model structure on simplicial sets and any fibrant object $Z \in \mathcal{K}$ the map $Z^L \to Z^K$ is a weak equivalence, then JM7(iii) holds.

Proof. The proofs of [Dug01, 3.2] for SM7 apply mutatis-mutandis to JM7. □

We use some observations of Hirschhorn, which can be found somewhere in his book. Firstly, if $\mathcal{M}$ is left proper, then so is the Reedy model structure on $\mathcal{M}^{\mathbb{\Delta}^{\mathrm{op}}}$ so we can apply Lemma 2.23. If $\mathcal{M}$ is combinatorial, then the Reedy model structure on $\mathcal{M}^{\mathbb{\Delta}^{\mathrm{op}}}$ is again so we can localize. By another observation of Hirschhorn, any (Reedy) fibration in the original model structure between fibrant objects in the localized model structure is still a fibration. So to recheck JM7(ii), by Lemma 2.23, we need only check that cotensoring with any simplicial set preserves the new fibrant objects (preservation of the old fibrations being obvious): This follows because taking products with simplicial sets preserve Joyal trivial cofibrations, the Joyal model structure being monoidal with all objects cofibrant. Then JM7(iii) will follow immediately by construction of the localization and Lemma 2.23.

It remains to show that the fibrant objects in the localized model structure are exactly the Rezk objects. It's clear that fibrant objects are complete Segal objects so it remains to show the converse. This is a bit subtle because we have to relate the two variable adjunction define weighted limits to the simplicial model structure but it can be done. The point is, by Reedy fibrancy, the desired lifting thing in simplicial sets is an isofibration between fibrant objects so lifting against an arbitrary trivial cofibration reduces to lifting against the specific ones mentioned above.

Remark. Rezk's model structure for complete Segal spaces (which we've chosen to call **Rezk spaces**) starts with the Quillen's simplicial model structure on **sSet** and then does the localization of Theorem 2.22 — but using a different tensor-cotensor-enrichment structure for bisimplicial sets. The difference between the tensors is that both are defined by restriction the cartesian product to some embedding of **sSet** into $\mathbf{sSet}^{\Delta^{op}}$ but in one the category of simplicial sets is embedded as constant simplicial objects (Rezk) and in the other as discrete simplicial objects (Verity). If I am understanding this correctly, the conclusion is that the model structure on bisimplicial sets for Rezk objects is enriched in one direction over Quillen's model structure and in the other direction of Joyal's model structure.

Basic category theory of quasi-categories. I should say something about how to do category theory with quasi-categories. Here I'm going to reflect my own personal bias and present things somewhat non-traditionally. This approach is joint work with Verity. I hope our papers will appear soon. All of our definitions of adjunctions, limits, and colimits and so on are the same as those of Joyal/Lurie but we think our approach makes it easier to generalize the proofs from standard category theory to the quasi-categorical context.

If I had to say something general about our strategy it would be that we come as far as possible through enriched category theory, which has the advantage of being already developed and not that hard to use. The philosophy of category theory is that important definitions can be encoded by conditions on maps, i.e., via universal properties, i.e., representably. So now you just have to write these definitions only referring to the hom-spaces (here) between two fixed objects and you've proven a theorem in enriched category theory.

So basically what we do is construct preferred models of things as weighted limits in simplicial sets. There's a general result, quite easy to prove, that says if the weights have a certain form (projectively cofibrant; i.e., built cellularly from representables) then if your diagram is of quasi-categories then the resulting weighted limit is again a quasi-category. Then we translate these simplicially enriched universal properties into (weak) 2-categorical universal properties and do the usual formal category theory.

This is a big story. I guess what I want to do now is tell a part of it I haven't yet talked about locally, which is to explain how to get the spaces to define universal properties representably and tell you some things that are true about them.

The definitions of adjunctions and limits and colimits make use of notion of a slice category so let's start by introducing the quasi-categorical analog. Given $B \xrightarrow{f} A \xleftarrow{g} C$ form

$$\begin{tikzcd} g \downarrow f \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A^{\Delta^1} \arrow[d] \\ B \times C \arrow[r, "f \times g"'] & A \times A \end{tikzcd}$$

It's a quasi-category with projections $C \xleftarrow{e_0} g \downarrow f \xrightarrow{e_1} B$ for evaluation at one or other end of the path. Furthermore, there's a canonical representative natural transformation $ge_0 \Rightarrow fe_1$

which I want to represent like this:

$$\begin{tikzcd} g \downarrow f \arrow[r, "e_0"] \arrow[d, "e_1"'] \arrow[dr, phantom, "\Downarrow\alpha"] & C \arrow[d, "g"] \\ B \arrow[r, "f"'] & A \end{tikzcd}$$

When I draw it in this way I'm actually thinking about just the homotopy class of the path in the homotopy category of the quasi-category $A^{g\downarrow f}$. These things are exactly 2-cells in the (strict) 2-**category of quasi-categories** which is obtained by taking homotopy classes of natural transformations and then forgetting all the higher dimensional cells in the hom-spaces between quasi-categories. It turns out this is a good place to make these definitions.

The point is that these comma quasi-categories are **weak comma objects**, meaning they satisfy a weak universal property. Given any simplicial set X and 2-cell

$$\begin{tikzcd} X \arrow[r, "d_0"] \arrow[d, "d_1"'] \arrow[dr, phantom, "\Downarrow\beta"] & C \arrow[d, "g"] \\ B \arrow[r, "f"'] & A \end{tikzcd}$$

there exists some $X \to g \downarrow f$ so that β factors along this map through α. Now these vertices in $g \downarrow f^X$ aren't unique but any two such are isomorphic (i.e., there's an isomorphism between them).

Note that this weak universal property is enough to determine the quasi-category $g \downarrow f$ up to equivalence. In the special case of this construction that will be relevant to the construction of limits and colimits, about more which in a moment, those of you who are more familiar with Lurie's "slicey" or "decalagey" description will be happy to know that those quasi-categories are equivalent to this one, which is the "fat slice" in that case and so satisfy the same universal property. But for definiteness, let us stick with this.

When g or f is an identity, we like to replace it with the name of the object. So for instance, given $f\colon B \rightleftarrows A\colon u$ we could form $f \downarrow A$ and $B \downarrow u$. Again these come with isofibrations to $A \times B$.

Definition 2.24. $f \dashv u$ is an **adjunction of quasi-categories** if and only if there is an equivalence $f \downarrow A \cong B \downarrow u$ over $A \times B$.

Note, because the pullbacks defining these quasi-categories are homotopy pullbacks, we can pull back this equivalence over vertices and get an equivalence $\mathrm{Hom}_A(fb, a) \simeq \mathrm{Hom}_B(b, ua)$ between mapping spaces for any $a \in A$ and $b \in B$.

Note also the right Quillen functor $(-)^X$ preserves everything we're talking about so we can see that adjunctions induce adjunctions between diagram categories. The same is true for precomposition though I'd prove this in a different way.

Using the equivalence and the universal property, the identity 2-cell at u can be used to define the counit, and the identity at f gives the unit, which in turn induce (possibly new) equivalences between the slice quasi-categories. There's a little bit of work here, but the point is we can do it all at once in more generality than I've just described the result.

Example 2.25. Take $A = \Delta^0$ so that f is the unique map and write t for u. What this says it that we have an equivalence $A \downarrow t \cong A$ over A. This $A \downarrow t$, by essentially the same geometry mentioned above but with the domain freed up is equivalent to $A_{/t}$ whose n-simplices are arbitrary $n+1$-simplices with last vertex t. Now the 2-of-3 property says that the projection

$A_{/t} \to A$ is a trivial fibration which says that it lifts against any sphere inclusion which says any sphere (bumping up dimensions) in A with last vertex t has a filler. This is to say that $t \in A$ is a **terminal object**. So we've shown that terminal objects are characterized by adjunction, just like in general category theory.

I'd like to say a bit about how general limits and colimits work. We begin with a general definition.

Definition 2.26. In a 2-category, an *absolute right lifting diagram* consists of the data

$$\begin{tikzcd} & C \arrow[d, "g"] \\ B \arrow[ur, "\ell"] \arrow[r, "f"'] \arrow[ur, phantom, shift right=3, "\Downarrow\lambda"] & A \end{tikzcd} \tag{2.27}$$

with the following universal property: given any 2-cell χ there exists a unique factorization as displayed below.

$$\begin{tikzcd} X \arrow[r, "c"] \arrow[d, "b"'] \arrow[dr, phantom, "\Downarrow\chi"] & C \arrow[d, "g"] \\ B \arrow[r, "f"'] & A \end{tikzcd} = \begin{tikzcd} X \arrow[r, "c"] \arrow[d, "b"'] \arrow[d, phantom, shift left=4, "\exists!\Downarrow"] & C \arrow[d, "g"] \\ B \arrow[ur, "\ell"] \arrow[r, "f"'] \arrow[ur, phantom, shift right=3, "\Downarrow\lambda"] & A \end{tikzcd}$$

Example 2.28. The counit of an adjunction $f \dashv u$ defines an absolute right lifting diagram

$$\begin{tikzcd} & B \arrow[d, "f"] \\ A \arrow[ur, "u"] \arrow[r, "\mathrm{id}_A"'] \arrow[ur, phantom, shift right=3, "\Downarrow\epsilon"] & A \end{tikzcd}$$

and, conversely, this data defines an adjunction.

Interpreting (2.27) in $\mathbf{qCat}_2$ permits us to form comma objects $C \downarrow \ell$ and $g \downarrow f$ with canonical cones as displayed.

$$\begin{tikzcd} C \downarrow \ell \arrow[r, "d_0"] \arrow[d, "d_1"'] \arrow[dr, phantom, shift left=2, "\Downarrow\gamma"] & C \\ B \arrow[ur, "\ell"'] & \end{tikzcd} \qquad \begin{tikzcd} g \downarrow f \arrow[r, "e_0"] \arrow[d, "e_1"'] \arrow[dr, phantom, "\Downarrow\alpha"] & C \arrow[d, "g"] \\ B \arrow[r, "f"'] & A \end{tikzcd} = \begin{tikzcd} g \downarrow f \arrow[r, "e_0"] \arrow[d, "e_1"'] \arrow[d, phantom, shift left=4, "\exists!\Downarrow"] & C \arrow[d, "g"] \\ B \arrow[ur, "\ell"] \arrow[r, "f"'] \arrow[ur, phantom, shift right=3, "\Downarrow\lambda"] & A \end{tikzcd}$$

Pasting the canonical cone under $C \downarrow \ell$ onto λ defines a map $C \downarrow \ell \to g \downarrow f$. The universal property of the absolute right lifting diagram applied to α defines a 2-cell under $g \downarrow f$ and over ℓ, displayed on the right above, which induces a map $g \downarrow f \to C \downarrow \ell$.

The following proposition makes two assertions. Firstly, the universal property of the absolute right lifting (ℓ, λ) implies these maps are equivalences. The second assertion is that if the map $C \downarrow \ell \to g \downarrow f$, definable without ascribing any universal property to λ, is an equivalence, then (ℓ, λ) defines an absolute right lifting diagram.

Theorem 2.29. *The data of* (2.27) *defines an absolute right lifting in* $\mathbf{qCat}_2$ *if and only if the induced maps form an equivalence* $C \downarrow \ell \simeq g \downarrow f$. *Conversely, any equivalence induces 2-cells as displayed above which can be used to define maps between comma objects which are again an equivalence.*

Definition 2.30. A **limit** of a diagram $d\colon X \to A$ is an absolute right lifting diagram

(2.31)
$$\begin{tikzcd} & A \arrow[d, "\mathrm{const}"] \\ \Delta^0 \arrow[ur, "\ell"] \arrow[r, "d"'] \arrow[r, phantom, shift left=3, "\Downarrow\lambda"] & A^X \end{tikzcd}$$

and conversely, or equivalently, it's an equivalence $A \downarrow \ell \cong \mathrm{const} \downarrow d$ over A.

Note the thing on the left-hand side has an obvious terminal object, namely the identity at ℓ, which passes across to a terminal object in the quasi-category of cones, which is the Lurie definition (in the equivalent "slicey" version).

A key advantage of this 2-categorical definition of (co)limits in any quasi-category is that it permits us to use standard 2-categorical arguments to give easy proofs of the expected categorical theorems.

Proposition 2.32. *Right adjoints preserve limits.*

Let's briefly recall the classical categorical proof. Given a diagram $X \stackrel{d}{\to} A$ and a right adjoint $A \stackrel{u}{\to} B$ to some functor f, a cone with summit b over ud transposes to a cone with summit fu over d, which factors uniquely through the limit cone. This factorization transpose back across the adjunction to show that the image of the limit cone under u defines a limit over ud.

Proof. Given an absolute right lifting diagram (2.31), an adjunction of quasi-categories $f \dashv u$, and hence an adjunction $f^X \dashv u^X$, we must show that

$$\begin{tikzcd} & A \arrow[r, "u"] \arrow[d, "c"] & B \arrow[d, "c"] \\ \Delta^0 \arrow[ur, "\ell"] \arrow[r, "d"'] \arrow[r, phantom, shift left=3, "\Downarrow\lambda"] & A^X \arrow[r, "u^X"'] & B^X \end{tikzcd}$$

is an absolute right lifting diagram. Given a cone

$$\begin{tikzcd} X \arrow[rr, "b"] \arrow[d] & & B \arrow[d] \\ \Delta^0 \arrow[r, "d"'] & A^X \arrow[r, "u^X"'] \arrow[u, phantom, "\Downarrow\chi"] & B^X \end{tikzcd}$$

we first transpose across the adjunction, by composing with f and the counit.

$$\begin{tikzcd} X \arrow[rr, "b"] \arrow[d] & & B \arrow[r, "f"] \arrow[d] & A \arrow[d] \\ \Delta^0 \arrow[r, "d"'] & A^X \arrow[r, "u^X"] \arrow[u, phantom, "\Downarrow\chi"] \arrow[rr, bend right, equal, "\Downarrow\epsilon^X"'] & B^X \arrow[r, "f^X"] & A^X \end{tikzcd} = \begin{tikzcd} X \arrow[r, "b"] \arrow[d] \arrow[r, phantom, shift right=4, "\exists!\Downarrow\zeta"] & B \arrow[r, "f"] & A \arrow[d] \\ \Delta^0 \arrow[urr, "\ell"] \arrow[rr, "d"'] \arrow[rr, phantom, shift left=3, "\Downarrow\lambda"] & & A^X \end{tikzcd}$$

Applying the universal property of the limit cone λ produces a factorization ζ, which may then be transposed back across the adjunction by composing with u and the counit.

$$\begin{tikzcd}
X \arrow[r, "b"] \arrow[d] \arrow[dr, phantom, "\exists!\Downarrow\zeta"] & B \arrow[r, "f"] \arrow[rr, bend left, equal] \arrow[rr, phantom, bend left, "\Downarrow\eta"] & A \arrow[r, "u"'] \arrow[d] & B \arrow[d] \\
\Delta^0 \arrow[urr, "\ell"] \arrow[rr, "d"'] & \arrow[r, phantom, "\Downarrow\lambda"] & A^X \arrow[r, "u^X"'] & B^X
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[rr, "b"] \arrow[d] \arrow[drr, phantom, "\Downarrow\chi"] & & B \arrow[r, "f"'] \arrow[rr, bend left, equal] \arrow[rr, phantom, bend left, "\Downarrow\eta"] \arrow[d] & A \arrow[r, "u"'] \arrow[d] & B \arrow[d] \\
\Delta^0 \arrow[r, "d"'] & A^X \arrow[r, "u^X"] \arrow[rr, bend right, equal] \arrow[rr, phantom, bend right, "\Downarrow\epsilon^X"] & B^X \arrow[r, "f^X"] & A^X \arrow[r, "u^X"'] & B^X
\end{tikzcd}$$

$$=
\begin{tikzcd}
X \arrow[rr, "b"] \arrow[d] \arrow[drr, phantom, "\Downarrow\chi"] & & B \arrow[rr, bend left, equal] \arrow[d] & & B \arrow[d] \\
\Delta^0 \arrow[r, "d"'] & A^X \arrow[r, "u^X"] \arrow[rr, bend right, equal] \arrow[rr, phantom, bend right, "\Downarrow\epsilon^X"] & B^X \arrow[r, "f^X"] \arrow[rr, bend left, equal] \arrow[rr, phantom, bend left, "\Downarrow\eta^X"] & A^X \arrow[r, "u^X"'] & B^X
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[rr, "b"] \arrow[d] \arrow[drr, phantom, "\Downarrow\chi"] & & B \arrow[d] \\
\Delta^0 \arrow[r, "d"'] & A^X \arrow[r, "u^X"'] & B^X
\end{tikzcd}$$

Here the second equality is immediate from the definition of η^X and the third is by the triangle identity defining the adjunction $f^X \dashv u^X$. The pasted composite of ζ and η is the desired factorization of χ through λ. The proof that this factorization is unique is left to the reader. It again parallels the classical argument: the essential point is that the transposes are unique. □

Fibrational perspective. To the best of my understand, the real innovation of the Lurie approach to quasi-category theory, extending the Joyal one, is his use of what might be called the "fibrational perspective," which involves a quasi-categorical generalization of the so-called "Grothendieck construction." This is a big story that I hope someone will talk about in more detail. Here let me just pave the way with some very elementary observations about the use of fibrations in model category theory and in quasi-category theory.

The following result implies that right derived functors of right Quillen functors can be constructed by precomposing with fibrant replacement.

Lemma 2.33 (Ken Brown's lemma)**.** *Any functor that preserves trivial fibrations between fibrant objects preserves weak equivalences between fibrant objects.*

Proof. In any model category, given any map $f\colon X \to Y$ between fibrant objects, it is possible to construct a fibrant object Z, fibrations

$$\begin{tikzcd}
& Z \arrow[dl, "p"] \arrow[dr, "q"] & \\
X \arrow[ur, bend left, "j"] \arrow[rr, "f"'] & & Y
\end{tikzcd}$$

and a section j to p that factors f as qj. When f is a weak equivalence, these maps are all weak equivalences, and the conclusion follows from the hypothesis by a straightforward application of the 2-of-3 property. □

Another way to think about fibrations is that they allow one to "avoid making choices."

Construction 2.34 (composition in a quasi-category)**.** We have a pushout in simplicial sets:[18]

$$\begin{tikzcd} \Delta^0 \arrow[r, "d^1"] \arrow[d, "d^0"'] & \Delta^1 \arrow[d] \\ \Delta^1 \arrow[r] & \Lambda^2_1 \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Homing into a quasi-category X, by adjunction, turns this pushout into a pullback

$$\begin{tikzcd} & & X^{\Lambda^2_1} \arrow[dl, two heads] \arrow[dr, two heads] \arrow[d, phantom, "\lrcorner", very near start] & & \\ & X^{\Delta^1} \arrow[dl, two heads, "s"'] \arrow[dr, two heads, "t"] & {} & X^{\Delta^1} \arrow[dl, two heads, "s"'] \arrow[dr, two heads, "t"] & \\ X & & X & & X \end{tikzcd}$$

all of whose maps are fibrations. We've relabeled the maps induced by $d^1, d^0 \colon [0] \rightrightarrows [1]$ as the source and target projections respectively. We think of $X^{\Lambda^2_1}$ as the quasi-category of composable arrows in X and the composite fibrations displayed as the various projections to the source, middle object, target, first factor, and last factor.

Now the combinatorics discussed above implies that the Segal map $X^{\Delta^2} \xrightarrow{\sim} X^{\Lambda^2_1}$ is a trivial fibration. In particular, there exists a non-canonically defined section which can be used to construct a composition map $X^{\Lambda^2_1} \xrightarrow{\circ} X^{\Delta^1}$ compatible with the source and target projections.

$$\begin{tikzcd} & X^{\Delta^2} \arrow[dl, two heads, "\sim"] \arrow[dr, "d^1"] & \\ X^{\Lambda^2_1} \arrow[ur, dashed, bend left] \arrow[d, two heads, "s"'] \arrow[drr, two heads, "t" near end] & & X^{\Delta^1} \arrow[d, two heads, "t"] \arrow[dll, two heads, "s"' near end] \\ X & & X \end{tikzcd}$$

$\boxed{\textbf{21/2}}$

3. Models of $(\infty, 1)$-categories: a bestiary (Clark Barwick)

The unification problem for higher homotopy theory is to build something so that

0-cells =	homotopy theories of $(\infty, 1)$-categories
1-cells =	equivalences of such
2-cells =	natural transformations
3-cells =	invertible modifications
4-cells =	invertible 4-morphisms

[18]This example shows that quasi-categories are not closed under colimits.

and so on. Here is the state of the art:

$$
\begin{tikzcd}
\mathbf{Cat}_T \arrow[r] & \mathbf{Cat}_S \arrow[r, dash, dashed] \arrow[dl, dash, dashed] \arrow[dd] & \mathbf{RelCat} \arrow[d] \arrow[dr] \arrow[ddr] & \\
\mathbf{Cat}_{A_\infty} \arrow[dr, dash, dashed] & & \mathbf{CSS} \arrow[d] \arrow[dr, shift left] & \mathbf{sSet}^+ \arrow[d] \\
& \mathbf{Seg}_f & \mathbf{Seg}_c \arrow[l] \arrow[r, shift left] & \mathbf{sSet}_{\text{Joyal}} \arrow[l, shift left] \arrow[ul, shift left]
\end{tikzcd}
\tag{3.1}
$$

Solid arrows are right Quillen equivalences; dashed arrows are equivalences of homotopy theories. A consequence of the result of Toën, generalized by the Barwick–Schommer-Pries theorem, is there are (essentially) no 2-cells in this diagram.

In the above we've written $\mathbf{Cat}_T$ for topologically enriched categories and $\mathbf{Cat}_S$ for simplicially enriched categories. Both[19] have appropriate model structures.

Theorem 3.2 (Bergner after Dwyer, Kan, Hirschhorn)**.** $\mathbf{Cat}_S$ *admits a model structure in which the weak equivalences are the DK-equivalences and the fibrant objects are the locally Kan simplical categories.*

RelCat is the category of **relative categories**, i.e., categories C equipped with a subcategory $wC \subset C$ of weak equivalences.

Theorem 3.3 (Kan)**.** *There is a model structure on* **RelCat***.*

Let us describe a weak equivalence $(C, wC) \to (\mathcal{D}, w\mathcal{D})$. There is a simplicial category $L^H(C, wC)$ attached to a relative category with the same objects and with hom-spaces $\mathrm{Map}_{L^H C}(x, y)$ defined to be some "hammock localization" colimit. A map of relative categories is a weak equivalence just when the associated simplicial functor, after applying L^H, is a DK-equivalence.

These three, i.e., the top line of (3.1), are all strict models of $(\infty, 1)$-categories. So why weaken? For one thing, none of these have a suitable internal hom; more precisely, none of these model structures are cartesian. For another thing, Mother Nature generates homotopy coherent diagrams, not necessarily strict diagrams (e.g. to understand delooping machines a la Jon Beck, Peter May, and Graeme Segal). Among the first to study this systematically were Boardman and Vogt.

Here $\mathbf{sSet}_{\text{Joyal}}$ is meant to denote the Joyal model structure on simplicial sets; **CSS** is the category of complete Segal spaces, with a model structure due to Rezk. There is a nerve functor $N\colon \mathbf{RelCat} \to \mathbf{CSS}$ (that is *not* the right Quillen functor of (3.1)) given by a doctored version of a construction of Rezk. Define $N(C, wC)\colon \mathbb{\Delta}^{\mathrm{op}} \to \mathbf{sSet}$ by

$$[m] \mapsto NwC^{[m]},$$

the "classifying nerve." The right-hand side is meant to be the ordinary nerve of the subcategory of pointwise weak equivalences between sequences of m composable arrows in C.

For $X \in \mathbf{CSS}$, we think of X_0 as being the moduli space of objects and X_1 as being the moduli space of morphisms. For $X = N(C, wC)$ we have $X_0 = NwC$ and $X_1 = Nw\mathrm{Arr}C$. Usually $N(C, wC)$ is *not* a fibrant object in **CSS**. However, by a lemma of Rezk $N(C, \mathrm{iso}C)$ is fibrant. (The hard part of the proof is Reedy fibrancy.)

[19]With point-set topological subtleties in the former case.

Proposition 3.4 (Barwick-Kan)**.** *If* (C, wC) *is a full sub relative category of a model category, then a Reedy fibrant replacement of* $\mathrm{Ex}^2 N(C, wC)$ *is a complete Segal space.*

This $\mathrm{Ex}^2 N$ is the right Quillen equivalence from relative categories to complete Segal spaces.[20]

Aside. This Ex^2 has a close relationship with the Thomason model structure on **Cat**, which comes with a right Quillen equivalence $\mathrm{Ex}^2 N\colon \mathbf{Cat} \to \mathbf{sSet}$.

Theorem 3.5 (Joyal-Tierney)**.** *There are two Quillen equivalences connecting complete Segal spaces and quasi-categories. One of them whose right Quillen functor is* $\mathbf{CSS} \to \mathbf{sSet}_{\mathrm{Joyal}}$ *defined by* $X \mapsto ([m] \mapsto \mathrm{Hom}(\Delta^0, X_m))$.

The proof has to do with two different Reedy model structures on bisimplicial sets, one where the enrichment is in the Quillen model structure and another where the enrichment is in the Joyal model structure.

We also have $\mathrm{Ex}^2 N\colon \mathbf{RelCat} \to \mathbf{sSet}^+$ where, using $\mathrm{sd}^2 \dashv \mathrm{Ex}^2$, we have to explain how to mark the edges in $\mathrm{sd}^2\Delta^n$ so as to define n-simplices to be $\mathrm{sd}^2\Delta^n \to (NC, wC)$.

$$\begin{tikzcd}
& & 1 \arrow[dl, "\sim"'] \arrow[dd] \arrow[dr] & & \\
& 01 \arrow[dr] & & 12 \arrow[dl, "\sim"'] & \\
& & 012 & & \\
0 \arrow[uur] \arrow[urr] \arrow[rr] & & 02 \arrow[u, "\sim"'] & & 2 \arrow[uul, "\sim"'] \arrow[ull, "\sim"'] \arrow[ll, "\sim"]
\end{tikzcd}$$

Note in this sd^2 the two subdivisions are *not* the same. For the second one, you turn around all the arrows. The point is that in the picture we've drawn, the outer triangle is equivalent to the triangle spanned by $0, 01, 012$. When we apply sd^2 all of the edges of this non-inverted triangle are "pushed inside" the picture.

Seg denotes the category of Segal categories. An early account, with a lot of ideas that are picked up later but developed "out of whole cloth," is Simpson's "Flexible sheaves."

Objects in the model category are simplicial spaces $X\colon \mathbb{\Delta}^{\mathrm{op}} \to \mathbf{sSet}$ so that X_0 is discrete.

Theorem 3.6 (Dwyer, Kan, Stover, Tamsamani, Simpson)**.** *This category has a model structure* $\mathbf{Seg}_c$ *whose cofibrations are monomorphisms, weak equivalences are DK-equivalences (in a suitable sense).*

The fibrant objects are the Segal categories (simplicial spaces with X_0 discrete and satisfying the Segal condition) that are additionally Reedy fibrant. (This last observation, precisely characterizing the fibrant objects, is due to Bergner.)

Why does this model categories weakly enriched in spaces? Given $a, b \in X_0$ define

$$\begin{tikzcd}
X(a,b) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_1 \arrow[d] \\
\{(a,b)\} \arrow[r] & X_0 \times X_0
\end{tikzcd}$$

[20]Note, we don't know what the fibrant objects are on **RelCat**. But the ones that arise as subcategories of model categories are well behaved: More precisely what you want is the 2-of-6 property and the three arrow calculus.

to be the mapping spaces. Given $a_0, \ldots, a_n \in X_0$ similarly define

$$\begin{tikzcd} X(a_0, \ldots, a_n) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_n \arrow[d] \\ \{(a_0, \ldots, a_n)\} \arrow[r] & X_0^{n+1} \end{tikzcd}$$

The zig-zag

$$X_1 \times_{X_0} X_1 \xleftarrow{\sim} X_2 \to X_1$$

pulls back to define

$$X(a,b) \times X(b,c) \xleftarrow{\sim} X(a,b,c) \to X(a,c).$$

Using this we can define a homotopy category of X to be the category with objects X_0 and hom-sets $\pi_0 X(a,b)$. Now the DK-equivalences for Segal categories are defined in exactly the way you would expect.

There is a variant model structure $\mathbf{Seg}_f$ due to Bergner designed to facilitate the comparison of (3.1). They have the same weak equivalences; the analogy is like between the projective and injective model structure.

$\mathbf{sSet}^+$ is the category of marked simplicial sets $(X, \mathcal{E})$ where $X_0 \subset \mathcal{E} \subset X_1$ and maps have to preserve the marked edges.

Theorem 3.7 (Street, Verity?)**.** $\mathbf{sSet}_+$ *admits a simplicial model structure whose cofibrations are monomorphisms and whose fibrant objects are* $(X, \mathcal{E})$ *where* X *is as quasi-category and* $\mathcal{E}$ *is the set of isomorphisms. Equivalences between fibrant objects are equivalences in the Joyal model structure.*

Remark. This isn't a complete list. For instance there are things you can do with the Thomason model structure on **Cat**.

Naturally arising examples.

- Model categories most naturally sit as relative categories.
- Cobordism categories (a la Galatius, Madsen), after a Reedy fibrant replacement, are complete Segal spaces.
- Homotopy coherent diagrams are quasi-categories (the original Boardman-Vogt example).
- Quillen's Q-construction is naturally a quasi-category (by work of Clark). Given an exact category $\mathcal{A}$, Quillen's Q-construction lets you build $Q\mathcal{A}$ with the same objects and where maps are equivalence classes of spans, where the one map is an admissible epimorphism and the other is an admissible monomorphism. But if you don't worry about whether composition is strictly defined, you can cut the "equivalence classes." Define a simplicial set instead and it will turn out to be a quasi-category. Now this thing ends up being the nerve of Quillen's $Q\mathcal{A}$, but this perspective allows you to generalize (dropping the conditions on the spans).
- Spaces give you A_∞-categories (its path space).

Closing remarks. As a final remark, the commutativity of the diagram (3.1), particularly using the dotted arrows, is not obvious. The result of Toën will be that these are all "orientation preserving" functors, commuting up to natural equivalence.

4. Toën's axiomatization of the homotopy theory of $(\infty, 1)$-categories (Gijs Heuts)

Goal 4.1. To find axioms a model category $\mathcal{M}$ should satisfy in order to be a **homotopy theory of** $(\infty, 1)$**-categories** that is

- satisfied by complete Segal spaces
- and so that the axioms characterize $\mathcal{M}$ up to equivalence.

A secondary goal should be to compute the automorphism space of such an $\mathcal{M}$.

Firstly, what about ∞-groupoids?

Fact 4.2. *The homotopy theory of ∞-groupoids is freely generated under homotopy colimits by $*$*

Aside. A generators-and-relations approach to $(\infty, 1)$-category theory naturally leads to complete Segal spaces. Starting with $\mathbb{\Delta}$, the free homotopy theory under homotopy colimits gives $\mathbf{sSet}^{\mathbb{\Delta}^{op}}$. What sort of relations should be imposed? Two relations

$$\Delta^1 \vee \cdots \vee \Delta^1 \xrightarrow{\sim} \Delta^n \qquad J \xrightarrow{\sim} \Delta^0.$$

This gives you complete Segal spaces (once you have Reedy fibrancy).

A final warm-up: How to characterize **Set**?

Giraud's axioms. What is a topos? It is a category C satisfying four axioms.

(i) C is (locally) presentable.
(ii) "Colimits are universal," i.e., colimits commute with pullbacks.
(iii) Coproducts are disjoint:

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d] \\ X \arrow[r] & X \coprod Y \end{tikzcd}$$

(iv) Equivalence relations are **effective**.

We should explain axiom (iv). An **equivalence relation** on X is $R \rightarrowtail X \times X$ so that, for any Y, $C(Y,R) \to C(Y,X) \times C(Y,X)$ is an equivalence relation on the set $C(Y,X)$. Given an equivalence relation, form the coequalizer

$$R \rightrightarrows X \longrightarrow X/R$$

The equivalence relation is **effective** if

$$\begin{tikzcd} R \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\ X \arrow[r] & X/R \end{tikzcd}$$

is a pullback.

On **Set**, which is of course a topos, all equivalence relations are effective.

Example 4.3. An open cover $\coprod_i U_i \to X$ gives an effective equivalence relation

$$\textstyle\coprod_{I,j} U_{i,j} \rightrightarrows \coprod_i U_i.$$

As this example suggests, the last axiom is crucial for the link to geometry.

Adding a fifth axiom:

(v) $\emptyset \to *$ is *not* an isomorphisms and $*$ is a generator (meaning $C(*, X) \to C(*, Y)$ is an isomorphism implies that $X \to Y$ is an isomorphism).

then the only C satisfying (i)-(v) is **Set**.

Model topoi/∞-topoi. Work on these ideals in the homotopical context has been done by Rezk, Toën, Vezzosi, Lurie and others.

Definition 4.4. A quasi-category (or model category) C is an ∞-**topos** if it satisfies

(i) C is presentable (combinatorial).
(ii) Homotopy colimits are universal.
(iii) Homotopy coproducts are disjoint.
(iv) Segal groupoid objects are effective.

Definition 4.5. A **Segal groupoid object** is a simplicial object X in C satisfying

(i) the Segal condition:
$$X_n \xrightarrow{\sim} X_1 \times^h_{X_0} \cdots \times^h_{X_0} X_1$$
is a weak equivalence for all $n \geq 2$.
(ii) groupoid axiom:
$$d_0 \times d_1 \colon X_2 \xrightarrow{\sim} X_1 \times_{X_0} X_1$$
is a weak equivalence. (This implies that "every morphism has a right inverse.")

When we write $|X|$ we mean a homotopy invariant geometric realization. (So if X is not Reedy fibrant, first perform a Reedy fibrant replacement.)

Definition 4.6. A Segal groupoid X is **effective** if

$$\begin{tikzcd} X_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_0 \arrow[d] \\ X_0 \arrow[r] & {|X|} \end{tikzcd}$$

is a homotopy pullback. Equivalently, the natural map from X to its Čech nerve
$$X \xrightarrow{\sim} \check{N}(X_0 \to |X|)$$
should be an equivalence, where
$$\check{N}(X_0 \to |X|)_n := X_0 \times_{|X|} \cdots \times_{|X|} X_0,$$
where X_0 appears in the iterated homotopy pullback $n + 1$ times.

Example 4.7. G a group in simplicial sets. Then you can form $BG_n = G^n$ and get $|BG| = BG$. This Segal groupoid object is effective

$$\begin{tikzcd} G \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d] \\ * \arrow[r] & BG \end{tikzcd}$$

i.e., $\Omega BG \simeq G$.

As before, we can add a fifth axiom:

(v) $\emptyset \to *$ is *not* an equivalence and $*$ is a generator, i.e., $f \colon X \to Y$ is and equivalence if and only if $\mathrm{Map}_C(*, X) \to \mathrm{Map}_C(*, Y)$ is a homotopy equivalence.

Claim 4.8. Any C satisfying (i)-(v) is equivalent to the category of spaces.

Our goal is now to try and generalize these axioms to get axioms that uniquely characterize the category of $(\infty, 1)$-categories.

Toward Toën's axioms. Note some of this can be done a bit more smoothly with ∞-categories in place of model categories, but we'll follow Toën's approach.

Definition 4.9. A model category $\mathcal{M}$ is **weakly cartesian closed** if

$$\mathrm{hocolim}_{i\in I}(x_i \times^h y) \xrightarrow{\sim} (\mathrm{hocolim}_{i\in I}\, x_i) \times^h y$$

Proposition 4.10 (Dugger)**.** *If $\mathcal{M}$ is combinatorial and weakly cartesian closed, then $\mathcal{M}$ is Quillen equivalent to a combinatorial, simplicial, cartesian closed model category in which all objects are cofibrant.*

Henceforth, we'll assume we're in this good setting (combinatorial, simplicial, cartesian closed) because it makes it easier to phrase the axioms. But it's important not to restrict to this setting only because interesting model categories (e.g. Bergner's simplicial categories) frequently fail to satisfy these hypotheses.

Definition 4.11. A **weak cocategory** is a cosimplicial object $C\colon \mathbb{\Delta} \to \mathcal{M}$ so that

$$C^1 \coprod_{C^0} C^1 \coprod_{C^0} \cdots \coprod_{C^0} C^1 \xrightarrow{\sim} C^n.$$

With such a thing we can define geometric realization, which we'll call C-**geometric realization**. Given $X \in \mathcal{M}^{\mathbb{\Delta}^{\mathrm{op}}}$, define

$$|X|_C := \int^{\mathbb{\Delta}^{\mathrm{op}}} C^n \times X_n,$$

where we should either assume that C is Reedy cofibrant or take the homotopy invariant coend (which amounts to taking a Reedy cofibrant replacement of C). This construction satisfies a universal property

$$\mathrm{Map}_{\mathcal{M}}(|X|_C, Y) \simeq \mathrm{Map}_{\mathcal{M}^{\mathbb{\Delta}^{\mathrm{op}}}}(X, \mathbb{R}\mathrm{Hom}_{\mathcal{M}}(C, Y)).$$

Definition 4.12. C is an **interval** if

- $C(0) \xrightarrow{\sim} *$
- $|J|_C \xrightarrow{\sim} *$.

To make sense of this, note we have a natural functor **Set** $\to \mathcal{M}$ given by $E \mapsto \coprod_E *$.[21] This gives a functor **sSet** $\to \mathcal{M}^{\mathbb{\Delta}^{\mathrm{op}}}$, which is how we understand J (the nerve of the free-standing isomorphism) as a simplicial object in $\mathcal{M}$.

We can also define the Čech nerve: Given $p\colon X \to Y$, define the homotopy pullback

$$\begin{tikzcd} X \times_Y^C X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{R}\mathrm{Hom}(C(1), Y) \arrow[d] \\ X \times X \arrow[r] & Y \times Y \end{tikzcd}$$

[21]You might want some cofibrancy hypotheses on $*$ in general, but here we're cartesian closed so everything's fine.

This would be the ordinary path space if we had used an ordinary interval. We then define $\check{N}^C(p)_n := X \times_Y^C \cdots \times_Y^C X$. We have homotopy pullbacks

$$\begin{tikzcd}
\check{N}^C(p)_n \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathbb{R}\mathrm{Hom}(C(n), Y) \arrow[d] \\
\mathbb{R}\mathrm{Hom}(C(0), X)^{n+1} \arrow[r] & Y \times \mathbb{R}\mathrm{Hom}(C(0), Y)^{n+1}
\end{tikzcd}$$

Note if you have a simplicial object X, there is a natural map

$$X \to \check{N}^C(X_0 \to |X|_C)$$

analogous to the situation before.

Definition 4.13. A simplicial object $X \in \mathcal{M}^{\mathbb{\Delta}^{\mathrm{op}}}$ is a **weak category** if

$$X_n \xrightarrow{\sim} X_1 \times_{X_0}^h \cdots \times_{X_0}^h X^1.$$

An object $X \in \mathcal{M}$ is 0-**local** if $\mathrm{Map}_{\mathcal{M}}(*, X) \xrightarrow{\sim} \mathrm{Map}_{\mathcal{M}}(C(1), X)$.

Morally the 0-local objects are the ∞-groupoids. At the moment this seems a bit weird because the notion of category was defined for simplicial objects and the definition of local was defined for objects, but we'll see in a moment how this makes sense.

Axioms 4.14. Now we state the axioms, given a model category $\mathcal{M}$ and $C\colon \mathbb{\Delta} \to \mathcal{M}$, which in practice is already given.[22]

(A1) For every fibrant, 0-local object $X \in \mathcal{M}$, $\mathcal{M}/X$ is combinatorial, weakly Cartesian closed. ("Hocolims are universal"—but only over 0-local objects.)
(A2) Given $\{X_i\}_{i\in I}$, $X = \coprod_i^h X_i$, then $X_i \times_X X_j = \emptyset$ if $i \neq j$. Also, for any Z, $\coprod_{i\in I}(Z \times_X X_i) \simeq Z$.
(A3) C is an interval.
(A4) For $X \in \mathcal{M}^{\mathbb{\Delta}^{\mathrm{op}}}$ a weak category so that X_0 and X_1 are 0-local, then

$$X \xrightarrow{\sim} \check{N}^C(X_0 \to |X|_C).$$

(The idea is that the interval C is somehow keeping track of direction, which it wasn't for groupoids.)
(A5) For X as in (A4) and a map $* \to |X|_C$, then

$$|X \times_{\mathbb{R}\mathrm{Hom}_{\mathcal{M}}} (C, |X|_C) * | \xrightarrow{\sim} *$$

where the $*$ on the left-hand side is a constant simplicial object.
(A6) A map $f\colon X \to Y$ is a weak equivalence if and only if $\mathrm{Map}_{\mathcal{M}}(*, X) \xrightarrow{\sim} \mathrm{Map}_{\mathcal{M}}(*, Y)$ and $\mathrm{Map}_{\mathcal{M}}(C(1), X) \xrightarrow{\sim} \mathrm{Map}_{\mathcal{M}}(C(1), Y)$
(A7) C is full and faithful, i.e.,

$$\mathrm{Hom}_{\mathbb{\Delta}}([n], [m]) \xrightarrow{\sim} \mathrm{Map}_{\mathcal{M}}(C(n), C(m)).$$

(A1)-(A3) should be thought of as Giraud axioms (i)-(iii). (A4) and (A5) are analogs of Giraud's (iv). Axiom (A6) says that the point and the interval are generators. (A7) is a non-degeneracy condition.

Claim 4.15. Complete Segal spaces satisfies (A1)-(A7).

[22]But note when you compute the moduli space, this cosimplicial object C is *not* treated as part of the structure.

Think of bisimplicial sets as simplicial spaces, then the first coordinate is the diagram coordinate and the second coordinate is the space coordinate.

Define $C(n)$ to be the copy of $\Delta^n \in \mathbf{sSet}^{\Delta^{op}}$ so that $\mathrm{Hom}(C(n), X) = X_n$. So $C(n)$ is the image of Δ^n under the "discrete space functor" $\mathbf{Set}^{\Delta^{op}} \to \mathbf{sSet}^{\Delta^{op}}$. 0-local means homotopically constant in the first variable (in the simplicial direction, aka the categorical direction, aka the direction of the Joyal model structure). Some of the axioms become tautological and others line up with the key results of Rezk's paper.

Uniqueness. More interesting is the fact that these axioms characterize a model category uniquely, and this is of course the reason for writing them down.

Theorem 4.16 (Toën)**.** *If $(\mathcal{M}, C)$ satisfies (A1)-(A7), then $\mathcal{M}$ is Quillen equivalent to complete Segal spaces.*[23]

Let's sketch the proof. We can simplify, assuming that $\mathcal{M}$ is combinatorial, simplicial, and that C is Reedy cofibrant. This allows us to build an adjunction between $\mathcal{M}$ and simplicial spaces. Using C, there is an adjunction

$$\mathbf{sSet}^{\Delta^{op}} \underset{S}{\overset{L}{\rightleftarrows}} \mathcal{M}$$

defined by $S(X)_n = \mathrm{Map}_{\mathcal{M}}(C(n), X)$ and $L(X) = \int^{\Delta^{op}} C(n) \otimes X_n$. When C is Reedy cofibrant, then $L \dashv S$ is a Quillen adjunction with respect to the Reedy model structure on bisimplicial sets. We have $L(J) \simeq |J|_C$ and, writing F for the cosimplicial object introduced above for simplicial spaces (as is Rezk's notation),

$$L(F(1) \coprod_{F(0)} \cdots \coprod_{F(0)} F(1) \to F(n))$$

is equivalent to

$$C(1) \coprod_{C(0)} \cdots \coprod_{C(0} C(1) \xrightarrow{\sim} C(0)$$

So by the universal property of Bousfield localization we get a Quillen adjunction between complete Segal spaces and $\mathcal{M}$.

Note that $\mathbb{R}S$ is conservative, i.e., reflects weak equivalences, by (A6). Now we use a standard trick: if $\mathbb{R}S$ is conservative, and the unit $1 \xrightarrow{\sim} \mathbb{R}S \circ L$ is a weak equivalence, then $L \dashv S$ is a Quillen equivalence.

So we want to prove $1 \xrightarrow{\sim} \mathbb{R}S \circ L$.

Proposition 4.17. *The left Bousfield localization of $\mathcal{M}$ with respect to $C(1) \to *$ is a model topos, in fact equivalent to the topos of spaces.*

This follows from two lemmas.

Lemma 4.18. *If $X \in \mathcal{M}^{\Delta^{op}}$ is a Segal groupoid, then $|X|_C \simeq |X|$.*

Lemma 4.19. *For every Segal groupoid X so that X_0 and X_1 are 0-local, then $|X|_C$ is 0-local.*

The proof of this second lemma uses the axiom (A5).

Sketch proof of proposition. (A1), (A2) $\Rightarrow$ Giraud's (i)-(iii). (A4), (A5) $\Rightarrow$ Giraud's (iv). Finally, (A6), (A7) $\Rightarrow \emptyset \neq *$ and $*$ generates. □

[23] The equivalence will turn out to be compatible with C by construction.

Using the proposition, we get an equivalence

$$\mathrm{Ho}(\mathbf{CSS})^{0-\mathrm{local}} \simeq \mathrm{Ho}(\mathcal{M})^{0-\mathrm{local}}.$$

Cleverly exploiting (A4) and this equivalence, you can "build up" every complete Segal space and check that

$$X \xrightarrow{\sim} \mathbb{R}S \circ L(X).$$

What is Aut(**CSS**)**?** From now on, because it's easier, write **CSS** for the associated quasi-category (the homotopy coherent nerve of the simplicial category on the fibrant-cofibrant objects in the model category). Define $\mathrm{Aut}(\mathbf{CSS}) \hookrightarrow \iota(\mathbf{CSS}^{\mathbf{CSS}})$ to be the subspace of self-equivalences in the maximal sub Kan complex.

$$\begin{tikzcd} \mathrm{Aut}(\mathbb{\Delta}/\mathbf{CSS}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Aut}(\mathbf{CSS}) \arrow[d] \\ * \arrow[r, "C"'] & \mathbf{CSS}^{\mathbb{\Delta}} \end{tikzcd}$$

Proposition 4.20. $\mathrm{Aut}(\mathbb{\Delta}/\mathbf{CSS}) \simeq *$

So if we fix a cosimplicial object, we have no automorphisms.

Proposition 4.21. *Every automorphism of* **CSS** *preserves* $\mathbb{\Delta}$*.*

Proof. An automorphism F necessarily preserves the terminal object. F also preserves the **gaunt** categories (categories in which every morphism is an identity), precisely because these correspond to 0-truncated objects. There is a characterization of these that is invariant under automorphism. It follows then that Δ^1 (the discrete simplicial space) is preserved: Δ^1 is characterized (for instance) by the existence of two distinct maps $* \rightrightarrows \Delta^1$, the fact that $\Delta^1 \neq * \coprod *$, and the only sub objects of Δ^1 are $\emptyset$, $*$, $* \coprod *$, Δ^1. It follows from $[n] \simeq [1] \coprod_{[0]} \cdots \coprod_{[0]} [1]$ that F fixes $\mathbb{\Delta}$. ◻

Proposition 4.22. $\mathrm{Aut}(\mathbf{CSS}) \simeq \mathbb{Z}/2$.

Proof. There is only one automorphism of $\mathbb{\Delta}$: it suffices to show that there is only one automorphism of

$$\begin{tikzcd} {[0]} \arrow[r, shift left=2] \arrow[r, shift right=2] & {[1]} \arrow[l] \end{tikzcd}$$

Furthermore $\mathrm{Aut}(\mathbb{\Delta}) \cong \mathbb{Z}/2$. Now we have

$$* \simeq \mathrm{Aut}(\mathbb{\Delta}/\mathbf{CSS}) \to \mathrm{Aut}(\mathbf{CSS}) \to \mathrm{Aut}(\mathbb{\Delta}) = \mathbb{Z}/2$$

that is surjective on π_0. This implies that $\mathrm{Aut}(\mathbf{CSS}) \simeq \mathrm{Aut}(\mathbb{\Delta})$. ◻

$\boxed{\textbf{7/3}}$

5. From n-fold Segal spaces to n-fold quasi-categories (Jeremy Hahn)

Before talking about (∞, n)-categories I'd better remind you about strict n-categories, at least in the case $n = 2$.

Definition 5.1. A (strict) **2-category** is a category enriched in categories.

There is a graphical calculus for working in such things (pasting diagrams).

Definition 5.2. A (strict) **double category** is a category internal to categories: a category of objects X_0, a category of arrows X_1 then maps $U\colon X_0 \to X_1$; $T, S : X_1 \to X_0$; and $\circ\colon X_1 \times_{X_0} X_1 \to X_1$ satisfying axioms.

Again, there is a graphical calculus. Arrows in X_0 are vertical arrows. Objects in X_1 are horizontal arrows. Morphisms in X_1 are squares. You can compose squares (just like you could compose 2-morphisms in a 2-category) in two directions, and there's a middle four interchange. This structure is symmetric; there's no axiomatic difference between the horizontal and vertical arrows.

Example 5.3. Every 2-category is a double category with no non-identity vertical arrows.

Example 5.4. Objects are model categories, horizontal morphisms are right Qullen arrows, vertical arrows are left Quillen functors, and squares are arbitrary natural transformations pointing down and to the left [Shu11].

Example 5.5. An object is a not-necessarily commutative ring. A horizontal arrow $A \to B$ is an (A, B)-bimodule, a vertical arrow is a ring homomorphism. Squares are bimodule maps.

Example 5.6. There is a double category nCob where an object is an $(n-1)$-manifold without boundary, a horizontal morphism is a cobordism, and a vertical morphisms is a diffeomorphism.

Definition 5.7. A strict ***n*-category** is a category object in category objects in ... in categories satisfying a further "globularity condition."

In the quasi-categorical context, this idea was first formalized by Barwick and it leads to the notion of an ***n*-fold Segal space**. Here is the idea: Suppose one already has a homotopy theory of n-fold homotopy theories $\mathcal{M}$. An $(n+1)$-fold homotopy theory is a simplicial object $X_\bullet \in \mathrm{Fun}(\mathbb{\Delta}^{\mathrm{op}}, \mathcal{M})$ satisfying the Segal conditions

$$X_m = X_1 \times^h_{X_0} X_1 \times^h_{X_0} \cdots \times^h_{X_0} X_1.$$

Starting with $\mathcal{M}$ the Kan complexes you get a Segal space. The completeness condition can be seen as one part of a globularity condition. So this gives you a complete Segal space. If we iterate this process you get a notion of an n-fold complete Segal space as a model for an (∞, n)-category.

Let's make this a bit more precisely. What category should an n-fold complete Segal space live in? Kan complexes live in simplicial sets. Complete Segal spaces live in bisimplicial sets. So an n-fold complete Segal space is a contravariant functor whose domain is $\mathbb{\Delta}^n$ and whose codomain is simplicial sets thought of with the Quillen model structure. We'll call objects of this category **$\mathbb{\Delta}^{\times n}$-spaces**. In Barwick's PhD thesis, he constructs a model structure on $\mathbb{\Delta}^n$-spaces whose fibrant objects are complete n-fold Segal spaces.

Model structure for *n*-fold complete Segal spaces. You can read about this is Barwick's thesis, in Lurie's Goodwillie paper, and also in [BSP12]. The starting point is the Reedy model structure on $\mathbb{\Delta}^{\times n}$-spaces. The Barwick model structure is a left Bousfield localization at three classes

- $\mathrm{Segal}_{\mathbb{\Delta}^{\times n}}$
- $\mathrm{Glob}_{\mathbb{\Delta}^{\times n}}$
- $\mathrm{Comp}_{\mathbb{\Delta}^{\times n}}$

The first localization gives you some sort of n-fold category. Localization at the second two classes together is what enforces the globularity condition.

It turns out it actually suffices to localize at maps between discrete functors, i.e., we can define all of these classes in $\mathbb{\Delta}^{\times n}$-**Set**.

Definition 5.8. Define $\square\colon (\mathbb{\Delta}\text{-}\mathbf{Set})\times(\mathbb{\Delta}^{\times n-1}\text{-}\mathbf{Set}) \to \mathbb{\Delta}^{\times n}\text{-}\mathbf{Set}$, cocontinuous in each variable, to be the external product of the representables

$$\Delta^{m_1} \square \Delta^{m_2,\ldots,m_n} = \Delta^{m_1,\ldots,m_n}.$$

Definition 5.9 (Segal maps)**.** $\mathrm{Segal}_{\mathbb{\Delta}}$ is the set of spine inclusions into Δ^m for $m \geq 0$. Then $\mathrm{Segal}_{\mathbb{\Delta}^{\times n}} = \{\mathrm{Segal}_{\mathbb{\Delta}} \square \Delta^{m_2,\ldots,m_n}\} \cup \{\Delta^{m_1} \square\ \mathrm{Segal}_{\mathbb{\Delta}^{\times n-1}}\}$.

Definition 5.10 (completeness maps)**.** $\mathrm{Comp}_{\mathbb{\Delta}} = \{* \to J\}$. Then $\mathrm{Comp}_{\mathbb{\Delta}^{\times n}} = \{\ \mathrm{Comp}_{\mathbb{\Delta}} \square \Delta^0\} \cup \{\Delta^{m_1} \square\ \mathrm{Comp}_{\mathbb{\Delta}^{\times n-1}}\}$.

Definition 5.11 (globularity maps)**.** Given $m_1, \ldots, m_n$ define

$$\hat{m}_i = \begin{cases} 0 & \text{if } \exists j < i \text{ s.t. } m_j = 0 \\ m_i & \text{otherwise.} \end{cases}$$

Then $\mathrm{Glob}_{\mathbb{\Delta}^{\times n}} = \{\Delta^{\hat{m}_1,\ldots,\hat{m}_n} \to \Delta^{m_1,\ldots,m_n}\}$.

Set-based models of (∞, n)-categories. We're going to define the notion of an n-fold quasi-category as a $\mathbb{\Delta}^{\times n}$-set. The first step is to define the notion of inner horns. Quasi-categories have lifts for the Segal maps (spine inclusions). This leads to the notion of an inner horn. Note that the Segal maps introduced above are maps of $\mathbb{\Delta}^{\times n}$-sets so we could start by asking if there's a model structure whose fibrant objects are the things that lift against this.

Definition 5.12. The class S of **inner anodyne** maps of $\mathbb{\Delta}^{\times n}$-sets

- consists of monomorphisms,
- is closed under pushouts and transfinite compositions,
- contains the n-fold Segal maps,
- is **spider saturated** or **transsaturated**.

Spider saturated means that given monomorphisms $A \to B \to C$ with $A \to B$ and $A \to C$ in S then $B \to C$ is too.[24]

Now a question is whether there is some set T containing the n-fold Segal maps $\mathrm{Segal}_{\mathbb{\Delta}^{\times n}}$ that is large enough so that its weak saturation is the inner anodyne maps?

Definition 5.13. Define

$$\Delta^{-,-,\cdots,-}\colon (\mathbb{\Delta}\text{-}\mathbf{Set})^{\times n} \to \mathbb{\Delta}^{\times n}\text{-}\mathbf{Set}$$

to be cocontinuous in each variable so that

$$\Delta^{\Delta^{m_1},\ldots,\Delta^{m_n}} = \Delta^{m_1,\ldots,m_n}.$$

Definition 5.14. Write $\boxtimes(f_1, \ldots, f_n)$ for the Leibniz tensor with respect to $\Delta^{-,-,\cdots,-}$. If the $f_i\colon x_i \to y_i$ are monomorphisms of simplicial sets, then

$$\boxtimes(f_1, \ldots, f_n) = \Delta^{y_1,x_2,\ldots,x_n} \cup \Delta^{x_1,y_2,x_3,\ldots,x_n} \cup \cdots \cup \Delta^{x_1,\ldots,x_{n-1},y_n} \to \Delta^{y_1,\ldots,y_n}.$$

Definition 5.15. A **generalized inner horn inclusion** is a map of the form

$$\boxtimes(\partial\Delta^{m_1} \to \Delta^{m_1}, \cdots, \Lambda^{m_k}_i \to \Delta^{m_k}, \cdots, \partial\Delta^{m_n} \to \Delta^{m_n})$$

which we will denote by $\Lambda^{m_1,\ldots,m_n}_{k,i}$ where $1 \leq k \leq n$ and $0 < i < m_k$.

For example $\boxtimes(\Lambda^2_1 \to \Delta^2, \partial\Delta^1 \to \Delta^1)$ is the thing that looks like the inclusion of a "trough" into a solid triangular prism.

---

[24]Joyal calls this the "right cancellation property."

Definition 5.16. An n**-fold quasi-category** is an n-fold simplicial set with lifts against the generalized inner horn inclusions.

The inner anodyne maps satisfy certain stability properties. Given two monomorphisms $f_1 \colon x_1 \to y_1$ and $g \colon x_2 \to y_2$ in $\mathbb{\Delta}^{\times n}$-sets define the pushout product $f \hat{\times} g$ using the product on $\mathbb{\Delta}^{\times n}$-**Set**.

Proposition 5.17.

- $\boxtimes(a_1, \ldots, a_n) \hat{\times} \boxtimes (b_1, \ldots, b_n) = \boxtimes(a_1 \hat{\times} b_1, \ldots, a_n \hat{\times} b_n)$
- $\mathrm{cell}(\boxtimes(S_1, \ldots, S_n)) \supset \boxtimes(\mathrm{cell}(S_1), \ldots, \mathrm{cell}(S_n))$

(∞, n)**-categories.** You can reverse the direction of the Glob maps. Since we're no longer talking about Bousfield localization but instead we're talking about lifting properties we want to turn these around and define $\mathrm{Glob}_{\mathbb{\Delta}^{\times n}} = \{\Delta^{m_1,\ldots,m_n} \to \Delta^{\hat{m}_1,\ldots,\hat{m}_n}\}$. Lifting against this says that there are no non-identity vertical arrows and other higher dimensional globularity conditions.

Definition 5.18. An (∞, n)-category is an n-fold quasi-category with lifts against all the maps in $\mathrm{Glob}_{\mathbb{\Delta}^{\times n}}$.

With respect to globular n-fold quasi-categories the lifting properties against the generalized inner horn inclusions are related to whiskering.

Completeness. Our notion of (∞, n)-category includes $\mathrm{Segal}_{\mathbb{\Delta}^{\times n}}$ and $\mathrm{Glob}_{\mathbb{\Delta}^{\times n}}$. What happened to $\mathrm{Comp}_{\mathbb{\Delta}^{\times n}}$? Note for $n = 1$ one you have lifts against the Segal maps you're already done (have a quasi-category). You don't need completeness (or globularity).

To explain what's going on, how would you say that a parallel pair of arrows $f, g \colon a \rightrightarrows b$ are equivalent? There is a **categorical equivalence** consisting of 2-cells $f \Rightarrow g$ and $g \Rightarrow f$ together with some condition relating their composites to the identities. In a quasi-category, there's also a **homotopical equivalence** given by gluing a 2-simplex (with other edge degenerate) in between f and g.

Does either sort of equivalence imply the other? Given a homotopical equivalence you can use the identity 2-arrow on f to define the domain of one of these generalized inner horn inclusions

$$\begin{tikzcd} & b \\ a \arrow[ur, bend left, "f"] \arrow[ur, "g"] \arrow[r, bend left, "f"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Downarrow\mathrm{id}"] & b \arrow[u, "\mathrm{id}"'] \end{tikzcd}$$

Because there is a filler in an $(\infty, 2)$-category we see that a homotopical equivalence implies a categorical equivalence.

But this implication isn't reversible. Imagine the strict 2-category on the picture

$$\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\Uparrow"] & \bullet \end{tikzcd} \tag{5.19}$$

where the 2-cells are inverses. The two 1-cells in the nerve of this thing will not be homotopically equivalent.

The intuition is that when the homotopical equivalences are the same as the categorical equivalences then the (∞, n)-category in question is **complete**. Note that nerves of strict n-categories are *not* complete (unless it's a *gaunt* n-category, in which case there are no categorical equivalences).

Remark. Something similar happens into complete Segal spaces. You need a nerve that takes the isomorphisms into account. Otherwise you don't get a *complete* Segal space.

Special outer horns. Another way to think about completeness uses special outer horns. In a quasi-category you can't expect to find a lift

$$\begin{tikzcd} & \cdot \\ \cdot \arrow[ur] \arrow[rr] & & \cdot \end{tikzcd} \hookrightarrow \begin{tikzcd} & \cdot \arrow[dr] \\ \cdot \arrow[ur] \arrow[rr] \arrow[r, phantom, "\sim"] & & \cdot \end{tikzcd}$$

unless this first diagonal arrow is an isomorphism. We'll make use of similar intuition in higher dimensions.

Definition 5.20. A **special outer horn** is a map of the form

$$\boxtimes(\partial\Delta^{m_1} \to \Delta^{m_1}, \ldots, \partial\Delta^{m_k} \to \Delta^{m_k}, * \to J, \emptyset \to *, \cdots, \emptyset \to *)$$

There are more things that you want to call special outer horns but this is a minimal collection so that if you're an (∞, n)-category and you lift against these then you lift against those things too. (∞, n)-categories that lift against all special outer horns are called **complete** (∞, n)**-categories**.

Homotopy theory of (∞, n)**-categories.**

Remark. Note we haven't yet presented a homotopy theory of (∞, n)-categories because we haven't discussed what it means for two such to be equivalence.

Theorem 5.21 (Hahn)**.** *There is a cartesian closed model structure on globular $\mathbb{\Delta}^{\times n}$-sets in which the cofibrations are the monomorphisms; fibrant objects are complete, globular, n-fold quasi-categories; an weak equivalences between (∞, n)-categories are the fully faithful and essentially surjective functors.*

Cartesian closure says that the internal hom $\mathrm{Hom}(X, Y)$ is the right thing (for instance, invariant under equivalence in both variables) so long as the target Y is complete. For intuition, there is some map to (5.19) from its completion that should be there but isn't.

Remark. Globular $\mathbb{\Delta}^{\times n}$-sets form a presheaf category on something that looks very similar to but isn't quite Θ_n. You could also put a model structure on Θ_n-sets but the horns aren't quite as easy to express.

This model structure is analogous to Tamsamani's except instead of iterating with the Quillen model structure on simplicial sets you iterate with the Joyal model structure. This model structure satisfies the unicity axiom of Barwick–Schommer-Pries. There are explicit Quillen equivalences with other models. This model is also very similar to Θ_n-sets. Furthermore, it's a complete Segal space iteration of a model that satisfies the axioms for $(\infty, 1)$. All of these are good settings to prove the unicity axioms.

What happens when $n \to \infty$**?** There is a model structure on $\mathbb{\Delta}^{\times\omega}$-sets given by localizing some Cisinski minimal model structure. There are two different philosophies about what a weak equivalence between ω-categories should be. One philosophy is that any ω-category with adjoints should be contractible. (Note that an (∞, n)-category with adjoints *is* contractible.) The basic question is whether a tower consisting of a pair of 1-cells pointing in opposite directions, 2-cells comparing their composites with the identities, 3-cells comparing these composites with identities, and so on has to eventually stop at identities in order to be called an equivalence or not.

Here a map of ω-categories is a **weak equivalence** if

(i) it is so in the model structure
(ii) it is fully faithful

(iii) it is essentially k-surjective for all $k \geq 0$.

Remark. Given a fibration $X \to Y$ between (∞, n)-categories, not necessarily complete. Then if the map is k-surjective for all k then it is a weak equivalence. Really the claim is that it's an acyclic fibration, so we're asked to prove that it lifts against all cofibrations. The idea is you can start by filling an inner horn part of the boundary in X. This filler isn't necessarily compatible in Y with the preexisting filler but these things combine to form an inner horn with a homotopy or something and so if you can lift the homotopy to X then you can solve the lifting problem. The details, using the fact that $X \to Y$ is a fibration, are a bit complicated.

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6. Conspectus of complicial sets (Dominic Verity)

Historical context. John Roberts defined complicial sets in the 1970s but without the context that appears today. In the 1980s, Ross Street put all this on firmer footing. More precisely, Ross defined a nerve functor from strict ω-categories to simplicial sets

$$N\colon \omega\text{-}\mathbf{Cat} \to \mathbf{sSet}.$$

The nerve is the right adjoint to an adjunction characterized by a cosimplicial object $O^\bullet\colon \mathbb{\Delta} \to \omega\text{-}\mathbf{Cat}$. The cosimplicial object is what Ross calls the **orientals functor $O^\bullet$**. On objects define

$$[0] \mapsto \bullet$$

$$[1] \mapsto \bullet \to \bullet$$

$$[2] \mapsto \begin{tikzcd} & \bullet \arrow[dr] & \\ \bullet \arrow[ur] \arrow[rr] \arrow[r, phantom, shift left=2, "\Uparrow"] & & \bullet \end{tikzcd}$$

$$[3] \mapsto \begin{tikzcd} & \bullet \arrow[r] & \bullet \arrow[dr] & & & & \bullet \arrow[r] \arrow[drr] & \bullet \arrow[dr] & \\ \bullet \arrow[ur] \arrow[urr] \arrow[rrr] & & & \bullet & \Rrightarrow & \bullet \arrow[ur] \arrow[rrr] & & & \bullet \end{tikzcd}$$

The orientation convention is "odd $\mapsto$ even": all the odd numbered codimension 1 faces are in the domain and all the even numbered codimension 1 faces are in the codomain. This determines the top level and everything in lower dimensions is defined compatibly with previous faces. The structure of the four simplex has to do with the MacLane pentagon.

From the combinatorial structure of a simplex you can build a **parity complex**. Ross defines what it means for structures like this to be loop free, defines freeness, and then presents these orientals as free structures on certain gadgets.

Via the ordinary nerve $N\colon \mathbf{Cat} \to \mathbf{sSet}$ categories sit as a full subcategory of simplicial sets. Quasi-category theory gets a lot of mileage out of this. Unfortunately, the corresponding nerve functor $N\colon \omega\text{-}\mathbf{Cat} \to \mathbf{sSet}$ is not fully faithful. The geometry of the cells is well encoded by the corresponding simplicial set but the composition structure is not encoded, precisely because you can't tell the difference between a picture in which the cell is a genuine non-trivial cell or a picture on which it is an identity. There are some aspects of the composition that you can recover but not everything.

John Roberts had only a vague sense of what ω-categories were and didn't have the nerve functor. Nonetheless he was able to describe the correct composition structure on simplicial sets to define complicial (=composition + simplicial) sets. The kind of extra structure he introduced on simplicial sets is called variously a **stratification**, **thinness**, or **hollowness**.

Formally, a **stratification** (X, tX), where $tX \subset X$ is just a subset (not a simplicial subset) of the simplicial set containing all the degeneracies. By convention tX doesn't contain any zero simplices. Roberts also asks that the thin 1-simplices are precisely the degenerate 1-simplices. There is a geometric idea behind this: if you assume a 1-simplex is thin, you've really crushed the $\bullet \to \bullet$ down to a point $\bullet$. But if you assume that a 2-simplex is thin, you haven't crushed entirely down to a $\bullet \to \bullet$.[25]

There's a canonical stratification on the image of the nerve which defines thin simplices to be things whose top dimensional cells are identities so now we have an adjunction

$$\begin{tikzcd} \omega\text{-}\mathbf{Cat} \arrow[r, shift right, "N"'] \arrow[r, phantom, "\perp"] & \mathbf{sSet}_t \arrow[l, shift right] \end{tikzcd}$$

writing $\mathbf{sSet}_t$ for the category of stratified simplicial sets (simplicial sets with a thinness structure). Note we can use thinness to recover composites of 1-cells in the image: $gf = h$ if there is some *thin* 2-simplex with boundary

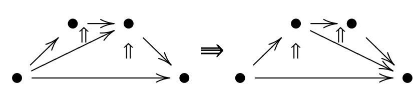
$$\begin{tikzcd} & \bullet \arrow[dr, "g"] & \\ \bullet \arrow[ur, "f"] \arrow[rr, "h"'] \arrow[rr, phantom, shift left=3, "="] & & \bullet \end{tikzcd}$$

It's a bit harder to see that this works in higher dimensions.

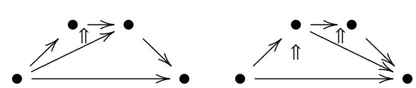
Ross conjectured that this nerve functor is fully faithful and some axioms introduced by Roberts, the **complicial sets**, characterize its image.

Theorem 6.1 (Verity)**.** *The nerve defines an equivalence of categories between ω-$\mathbf{Cat}$ and the full subcategory of complicial sets in $\mathbf{sSet}_t$.*

A $\Lambda^{3,1}$-horn of



is missing the top dimensional simplex and the face opposite the vertex 1



The sort of question we're asking is under what conditions in the nerve of a strict ω-category would you be able to find a 2-cell to fill the missing part so that the left-hand and right-hand composites agree. The only information we're recording is the identities, so you can deduce the condition: if the 3rd face is a thin simplex, then there is a unique filler for the missing 1st face so that the resulting 3-simplex is thin.

So the lifting condition is that if you're given a horn $\Lambda^{3,1} \to NC$ so that the 3rd face is thin, there is a unique extension to a thin 3-simplex. This sort of thing is precisely a lifting property against a particular map of stratified simplicial sets with underlying map $\Lambda^{3,1} \to \Delta^3$ of simplicial sets.

Making this precise in general has to do with certain cocycle conditions introduced by Ross in his paper [Str87]. We won't talk about that because we actually can get by with a bit less.

Definition 6.2. Let $\Delta^{n:k}$ (formerly $\Delta_k[n]$) denote the k-**admissible** n-simplex, which has Δ^n as its underlying simplicial set. Declare $\alpha\colon [r] \to [n]$ to be thin if and only if $\{k-1, k, k+1\} \cap [n] \subset \operatorname{im}\alpha$.

---

[25]Marked simplicial sets would be an example were it not for this last condition, and indeed it will disappear when we start talking about weak things.

Remark (Mike). If you draw the three vertices $k-1, k, k+1$ on a line (as opposed to spanning a triangle) then the thin faces are precisely the things that span a smaller dimension than you'd expect.

Note we only have three non-thin codimension 1 faces. The two omitting $k-1$ and $k+1$ are interpreted as having composite the face omitting k. Write $\Lambda^{n:k} \subset \Delta^{n:k}$ for the k-horn with the same thin structure.

Lemma 6.3 (Street)**.** *You have unique extensions*

$$\begin{tikzcd} \Lambda^{n:k} \arrow[r] \arrow[d] & NC \\ \Delta^{n:k} \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

There are a couple extra variants which have to do with composing thin simplices: if you compose two thin $(n-1)$-simplices the composite should again be thin. This can be encoded by lifting against some maps $\Delta^{n:k+} \to \Delta^{n:k*}$. Note any extensions

$$\begin{tikzcd} \Delta^{n:k+} \arrow[r] \arrow[d] & NC \\ \Delta^{n:k*} \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

are automatically thin.

Definition 6.4 (Roberts)**.** A **complicial set** is a stratified simplicial set X satisfying the unique extension condition against these maps.

What's really remarkable is that he made this definition in the 1970s without reference to any of the nerve or ω-category stuff.

Weak things. Of course now we're more interested in weak variants of ω-categories. How do we weaken? Firstly, drop the condition that the only thin 1-simplices are degenerate because we now think of thin things as being equivalences and not just identities. When we're thinking about thin things as equivalences, of course it's ridiculous to ask for unique extensions. Henceforth, by "complicial set" we mean **weak complicial sets** which will be a model for (∞, ∞)-categories: at every level we have cells that might not be equivalences. The things we know definitely behave like equivalences are the things we marked as thin.[26]

To get anywhere we have to talk about the **Gray tensor product**. Given a pair of 2-categories $\mathbf{C}, \mathbf{D}$, we can form a 2-category $[\mathbf{C}, \mathbf{D}]_p$ of strict functors $F \colon \mathbf{C} \to \mathbf{D}$, pseudo natural transformations, and modifications. The usual product of 2-categories isn't adjoint to this 2-category but there is a tensor product so that we have a two-variable 2-adjunction

$$\mathbf{B} \to [\mathbf{C}, \mathbf{D}]_p \qquad \mathbf{B} \otimes \mathbf{C} \to \mathbf{D}.$$

The tensor product is named after its inventor John Gray.

It's not too complicated to write down but it takes a bit of time. Instead we'll describe examples:

$$(\bullet \to \bullet) \otimes (\bullet \to \bullet) = \begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\cong"] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

[26]Note however there might be more equivalences; this will have to do with saturation later.

There's also a version of the Gray tensor product $\otimes_\ell$ for the 2-category replacing pseudo natural transformations by lax natural transformations. As you'd expect:

$$(\bullet \to \bullet) \otimes_\ell (\bullet \to \bullet) = \begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\Uparrow"] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

What these Gray tensor products encode are all sorts of geometric facts about cubes. These are both nice 2-monoidal products, though only $\otimes$ is symmetric. There are nice relations between Gray tensor products and braid groups: the 3rd Reidermeister move is a 2-categorists presentation of the geometry of the cube.

The Gray tensor product can extend to higher dimensions. Without worrying too much about orientation, the lax tensor product of a 1-cell with a 2-cell looks like a cylinder with 2-cells at either end and two square faces, each with 2-cells sitting inside them, and then a 3-cell sitting between the two possible composites of the end 2-cells with the 2-cells in the side squares. Crans showed in the early 90s that this idea can be extended to define a Gray tensor product on ω-categories.

In the stratified context the analog of the Gray tensor product is the thing that allows us to build arbitrary products and arbitrary cylinders from the simplices. I'm going to try and explain $\Delta^n \otimes \Delta^m$. This product will preserve colimits in each variable so this defines the entire thing. By a leap of faith assume that on underlying simplicial sets $\Delta^n \otimes \Delta^m$ is just the usual cartesian product. (There is a strong intuition for this based on the relation between the Gray tensor product on ω-categories and the ordinary product.) The question is how to define the thin simplices.

Draw $\Delta^n \times \Delta^m$ as the nerve of the ordered set $[n] \times [m]$ so you get some kind of grid where $[n]$ goes across and $[m]$ goes up:

$$\begin{tikzcd}
\bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \\
\bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[u] \\
\bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[u] \\
\bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[u] \\
\bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[u] \\
\bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[u]
\end{tikzcd}$$

Note that a dimension of a cell in the product is the sum of the dimensions of the two components. The stuff in here of dimension $n + m$ are called **shuffles**. Remember you're thinking of simplices as oriented from odd to even. Here things are facing in a bunch of oblate directions so they aren't really composable (unless say you're in a complex). We want to make everything thin so that there's just one thing that's a genuine $n + m$-dimensional oriented cell. One choice is to pick the shuffle that goes along the bottom and then up the right-hand edge and make all the other shuffles thin.

This explains what to do with the shuffles but what about the lower simplices. Make anything thin that at some point goes directly up and then directly to the right. The reason is that these simplices correspond to composites of other stuff—these are called **mediator simplices**. Note all of the thin shuffles are themselves mediator simplices.

To fully characterize this tensor product we also need to consider products of thin simplices. Write $\Delta^{n\sharp}$ for the n-simplex with top dimensional face thin. In $\Delta^{n\sharp} \otimes \Delta^m$ you also need anything that projects down onto the thin face $\Delta^{n\sharp}$ to also be thin. Extending by colimits we have

$$\mathbf{sSet}_t \times \mathbf{sSet}_t \xrightarrow{\otimes} \mathbf{sSet}_t$$

and furthermore this product is closed on both sides. The right adjoints correspond to either lax natural transformations or oplax natural transformations $- \otimes X \dashv \mathrm{lax}_r(X, -)$ and $X \otimes - \dashv \mathrm{lax}_l(X, -)$. The only problem is that these products are monoidal: the associativity isomorphism of underlying simplicial sets doesn't preserve thinness. Unfortunately, fixing the associativity breaks the closed structure at least on all of $\mathbf{sSet}_t$. Eventually the solution is to cut down to a subcategory on which these things coincide. Counterexamples to associativity are pretty simple to find: consider $\Delta^1 \times \Delta^1 \times \Delta^1$.

This other tensor product $\otimes'$ has to do with subdivision. For example, consider the diagonal 5-simplex in $\Delta^5 \times \Delta^5$.

$$\begin{tikzcd}
\bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \\
\bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] \arrow[ur, dashed] & \bullet \arrow[u] \\
\bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] \arrow[ur, dashed] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[u] \\
\bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] \arrow[ur, dashed] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[u] \\
\bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] \arrow[ur, dashed] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[u] \\
\bullet \arrow[r] \arrow[u] \arrow[ur, dashed] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[r] \arrow[u] & \bullet \arrow[u]
\end{tikzcd}$$

The idea is this should be thought of as the composite of the six 5-simplices spanning the six boxes containing the top-left vertex and then having diagonally opposite vertex on this line.

On a 3-simplex this subdivision goes like this. Take a plane parallel to the 123-face and slice halfway between 0 and this face. Then do the same thing for the 012-face and the vertex 3. Now to partition between 1 and 2 take the pair of edges 01 and 23 and slice along the plane parallel to the one determined by these two edges that lies halfway between 0 and 1. The result after this slicing is two 3-simplices including the vertices 0 and 3 and two triangular prisms containing 1 or 2 as one of the vertices.

This leads to another definition of the thinness structure on the product $\Delta^n \times \Delta^m$, which says that something is thin if and only if its a composite in this sense of special shuffles all of which are thin.

We have an inclusion $X \otimes Y \to X \otimes' Y$ and this is a composite of pushouts of things which say that if something is a composite of all thin things its again thin. So on the subcategory where you have extensions of this form (along the $\Delta^{n:k+} \to \Delta^{n:k*}$) then these two tensor products ($\otimes$, the closed one, and $\otimes'$, the associative one) coincide. This is true on the subcategory of **precomplicial sets**.

$\boxed{\textbf{28/3}}$

7. Conspectus of complicial sets continued (Dominic Verity)

Today we're going to introduce complicial sets of simplex bordisms.

Simplex bordisms. You could imagine a 2-dimesional manifold with boundary thought of as a bordism embedded in a 3-simplex with boundary embedded in its boundary. The boundary components are allowed to intersect with the 1-skeleton of the 3-simplex. Its faces will then have embedded 1-manifolds whose boundary lies on the boundary of the 1-simplex. This is essentially the same thing as a **stratified manifold with boundary**, except that is structured around the combinatorics of the cube, while this is structured around the combinatorics of the simplex.

Let's make this precise. I'm going to elide mention of piecewise linear everywhere. We'll define a **simplex bordism** to be a functor $M\colon \mathbb{\Delta}_f/[n] \to \mathbf{Man}_e$. Given a monomorphism $\alpha\colon [r] \to [n]$, you get a manifold M^α. Associated to a morphism $[s] \xrightarrow{\delta} [r]$ from $\alpha\delta$ to α you get an embedding $M^{\alpha\delta} \to M^\alpha$. We impose some axioms:

(i) For each $\delta^i\colon [r-1] \to [r]$ and the associated $M^{\alpha\delta^i} \to M^\alpha$, either $M^{\alpha\delta^i}$ is empty or it is a codimension-1 subspace landing in the boundary ∂M^α.
(ii) For all $x \in M$ (also the image of $\mathrm{id}_{[n]}$), there is a unique α so that $x \in \mathrm{int}(M^\alpha)$.

Note the dimension of M could be arbitrary; it doesn't have to relate to n.

Let $\underline{\mathrm{Cob}}$ denote the set of homeomorphism classes of simplex bordisms. It's a graded set by the $[n]$'s, a semi-simplicial set by precomposition, and can also be shown to be a simplicial set by defining degeneracy operators appropriately. In fact, $\underline{\mathrm{Cob}}$ is a Kan complex: Write $H_m(i_1, \ldots, i_k) \subset \Delta^m$ for the subset generated by the faces listed. Note that a map of simplicial sets $X \to \underline{\mathrm{Cob}}$ is exactly a functor $\mathbf{el}(X) \to \mathbf{Man}_e$. Write $\cup_{i=1}^k F$ for the colimit of the diagram $F\colon \mathbf{el}H_n(i_1, \ldots, i_k) \to \mathbf{Man}_e$. Some inductive argument can then be used to build the necessary manifolds that show horns have fillers: Take the union of all the faces, then the product of this with the interval. Glue one end of this to the original manifold sitting over the given horn and identify the other end with the manifold on the new face. (As an aside, any semisimplicial Kan complex can be promoted to a simplicial set, that's still a Kan complex.)

Imagine a horn Λ^2_1 with three points sitting on each edge. There are two fillers, one of which has six points on the "composite" edge, and another of which has non. Both 2-simplices have a (non-connected) 1-manifold sitting over their interiors. Now these are themselves cobordant, filling some 3-dimensional horn, but we'd like another way to distinguish them. We'll use a stratification.

Stratification. We'll use the same convention from Street's orientals: orienting simplices from their odd faces to their even faces. Given $M\colon \mathbb{\Delta}_f/[n] \to \mathbf{Man}_e$, write $\partial_- M$ for the union of the odd faces and $\partial_+ M$ for the union of the even faces. We have $\partial M = \partial_- M \cup \partial_+ M$. Declare M to be **thin** if and only if M collapses to $\partial_- M$. In fact, in this case, M shells (like collapse but we insist that at each step M remains a manifold, and not just a polytope) to $\partial_- M \times I$.

As it turns out the construction of canonical fillers used to prove that $\underline{\mathrm{Cob}}$ is a Kan complex also shows that it is a weak complicial set. An important part of this story is the "boundary reorganization lemma" which says that you can permute the boundaries of thin simplices (and overcome the reference to "odd" and "even" in the definition of the thinness structure).

Stratified structures. Is the notion of stratification intrinsic to the simplicial set? I.e., is there a unique stratification that makes a given simplicial set a complicial set. The answer is definitely no.

For instance, in the example above, there are more general notions of thinness that we'd prefer—say invertible cobordisms as opposed to just trivial cobordisms.

Theorem 7.1. *Every weak complicial set is equivalent in the model structure to the homotopy coherent nerve of something enriched over weak complicial sets. Furthermore,* *the homotopy coherent nerve of something enriched in weak complicial sets thin above dimension n is thin above dimension $n+1$.*

The dimension shift allows for an inductive argument: given a complicial set, expand the stratification by throwing in everything you think ought to be equivalences and that again is a weak complicial set. (Proof: "ought to be equivalences" is a representable notion.)

Q. Is there a maximal stratification?

Let's say a bit more about this. At dimension 1, it's fairly straightforward: a 1-simplex ought to be invertible if it extends to J, say, or has an inverse, or whatever. At dimension 2, given

$$\begin{tikzcd} & \bullet \arrow[dr] & \\ \bullet \arrow[ur, "x"] \arrow[rr, "y"'] & & \bullet \end{tikzcd}$$

you might declare it to be invertible if there is

$$\begin{tikzcd} & \bullet \arrow[dr] & \\ \bullet \arrow[ur, "y"] \arrow[rr, "x"'] & & \bullet \end{tikzcd}$$

so that certain horns you can build from these faces have fillers. In particular, this implies that the unlabeled edges are also invertible. Calling this edge z, it turns out you can get, from a filling argument, a 2-simplex

$$\begin{tikzcd} & \bullet \arrow[dr, equal] & \\ \bullet \arrow[ur, "yx"] \arrow[rr, "z"'] & & \bullet \end{tikzcd}$$

that should be thought of as invertible.

Writing $\underline{\mathrm{Cob}}_t$ for the original weak complicial set and write $\underline{\mathrm{Cob}}_i$ for the **saturation**, the thing in which everything that looks like an equivalence is an equivalence. By some abstract nonsense this is again a weak complicial set. Furthermore, there is another model structure whose fibrant objects are saturated weak complicial sets and this is fibrant replacement in that model structure. (In particular $\underline{\mathrm{Cob}}_t$ and $\underline{\mathrm{Cob}}_i$ are equivalent, which is not true in the original model structure.) In this case, though not in general, the fibrant replacement can be done without changing the underlying simplicial set.

Another example of a non-saturated thing is the (weak) complicial set that arises as the nerve of a 2-category with identities marked. If we instead give it the stratification with equivalences marked, we have a map of weak complicial sets from the former to the latter, but it's not an equivalence in the original model structure. It is though in the new model structure. Thus, the model structure for ∞-categories is the one where the fibrant objects are the saturated weak complicial sets, not the one that's actually in the paper [Ver08b].

Q (Mike)**.** Something about loop spaces and bordisms where you restrict to manifolds of a fixed dimension.

There's also $\underline{\mathrm{Cob}}_k$ which has the stratification that comes from being a Kan complex (a Kan complex is a weak complicial set in which everything above dimension 0 is thin). Interestingly, $\underline{\mathrm{Cob}}_i$, the saturation of $\underline{\mathrm{Cob}}_t$, is not $\underline{\mathrm{Cob}}_k$, i.e., not everything is thin. How do you prove this? Define *another* stratification $\underline{\mathrm{Cob}}_h$ where M is thin if $\partial_- M \hookrightarrow M$ is a homotopy equivalences (a simplicial h-cobordism). Again this is a weak complicial set. By the Eilenberg-Mazur swindle it is stratified. It contains $\underline{\mathrm{Cob}}_t$ but is smaller than $\underline{\mathrm{Cob}}_k$.

There are lots of analogs of this sort of construction in other contexts. The only significant bound on a sequence of such inclusions is if all simplices above a given dimension are thin.

Upshot: weak complicial sets with everything thin above dimension n are (∞, n)-categories. Weak complicial sets with non-thin things at each level are (∞, ∞)-categories and this discussion suggests that these are genuinely different things.

Homotopy coherent nerve. The homotopy coherent nerve is a functor $N\colon \mathbf{Strat}\text{-}\mathbf{Cat} \to \mathbf{Strat}$. This is defined via a functor $\mathbb{\Delta} \to \mathbf{Strat}\text{-}\mathbf{Cat}$ with $[n] \mapsto \mathcal{O}^n$. Define $\mathcal{O}^n$ to be the simplicial category whose objects are $[n]$ and define $\mathcal{O}(i, j) \cong (\Delta^1)^{j-i-1}$, so that this is the simplicial object defining the ordinary homotopy coherent nerve. It remains only to give it its stratification and for this we use the stratification of $(\Delta^1)^{\otimes(j-i-1)}$. There's a bit of subtlety between the tensor products of the enrichment and the associative one but you can reflect back to where you need to go.

8. Models for the theory of $(\infty, 1)$-operads (Gijs Heuts)

I could go on for hours about this. Today I'll try and focus on the three most common models. Convention: "operad" means symmetric and colored.

Fix a symmetric monoidal category $\mathcal{V}$. An operad O in $\mathcal{V}$ consists of

- a set C of colors (objects)
- for every $c_1, \ldots, c_n, d \in C$ and object $O(c_1, \ldots, c_n; d) \in \mathcal{V}$, which we think of as parametrizing operations with inputs $c_1, \ldots, c_n$ and output d
- $I \to O(c; c)$ for all c, where I is the monoidal unit
- Σ_n-actions $O(c_1, \ldots, c_n; d) \to O(c_{\sigma(1)}, \ldots, c_{\sigma(n)}; d)$ for each $\sigma \in \Sigma_n$
- composition maps

$$O(d_1, \ldots, d_n; e) \otimes O(\vec{c_1^i}; d_1) \otimes \cdots \otimes O(\vec{c_n^j}; d_n) \to O(c_1^1, \ldots, c_n^{m_n}; e)$$

satisfying axioms. If $C = \{c\}$ then $O(c, \ldots, c; c) =: O(n)$ if there are n-copies. Note we allow nullary operations; i.e., the list of inputs can be empty. Operads in this sense are also called **symmetric multi categories**.

Example 8.1. If C is a category, we get an operad $j_!C$ in sets whose colors are the objects of C. Define

$$j_!C(c_1, \ldots, c_n; d) = \begin{cases} C(c, d) & n = 1 \\ \emptyset & \text{else.} \end{cases}$$

This is part of an adjunction

$$\begin{tikzcd} \mathbf{Cat} \arrow[r, shift left, "j_!"] \arrow[r, phantom, "\perp"] & \mathbf{Op}_{\mathbf{Set}} \arrow[l, shift left, "j^*"] \end{tikzcd}$$

where j^* discards all non-unary operations.

Example 8.2. If $(\mathcal{D}, \otimes)$ is symmetric monoidal you get an operad $\underline{\mathcal{D}}$ with colors the objets of $\mathcal{D}$ and with

$$\underline{\mathcal{D}}(c_1, \ldots, c_n; d) := \mathcal{D}(c_1 \otimes c_n; d).$$

Today we're only interested in operads taking values in sets or in simplicial sets. Write **sOp** for the latter category.

Theorem 8.3 (Cisinski-Moerdijk)**.** **sOp** *carries a model structure that extends the Bergner model structure on simplicial categories. A weak equivalence is a map that's a weak homotopy equivalences on the mapping spaces and so that the map between the homotopy categories of the simplicial categories of unary morphisms (applying j^*) are essentially surjective.*

Q. So simplicial operads gives one model for $(\infty, 1)$-operads. Why do we want others?

- When working with **qCat**, **CSS**, or **SegCat** want models that are adapted to these.
- Cofibrancy is hard to achieve (non-examples: commutative operad, $\mathbb{E}_n$ operads). Other models will have a larger class of cofibrant objects
- Tensor products: **sOp** carries a (symmetric) Boardman-Vogt tensor product $P \otimes_{BV} Q$ characterized by the property that $\mathrm{Alg}_{P\otimes Q}(C) \simeq \mathrm{Alg}_P(\mathrm{Alg}_Q(C))$, but **sOp** is not a monoidal model category.[27]

There will be a monoidal structure on the homotopy category that does what you want.

Example 8.4. $\underline{\mathrm{Ass}} \otimes \underline{\mathrm{Ass}} \simeq \underline{\mathrm{Com}}$ and $\mathbb{E}_1 \otimes \mathbb{E}_1 \simeq \mathbb{E}_2$.

We'll focus on two models:

- dendroidal sets (Moerdijk, Weiss, Cisinski)
- ∞-operads

Both are adapted to (marked) simplicial sets with the Joyal model structure.

Dendroidal sets. Write N: **sCat** $\to$ **sSet** for the homotopy coherent nerve. We want

$$\begin{tikzcd} \mathbf{sCat} \arrow[d, "N"'] & \mathbf{sOp} \arrow[l, "j^*"'] \arrow[d, dashed, "N_d"] \\ \mathbf{sSet} & \mathbf{dSet} \arrow[l, dashed, "i^*"] \end{tikzcd}$$

i.e., we want a category that admits a "nerve of operads" functor.

We'll enlarge $\mathbb{\Delta}$ to $\mathbb{\Omega}$, the category of trees.[28] Objects are finite rooted trees (meaning there's a single edge—a "leaf"—sticking out the bottom) that do *NOT* have a planar structure (though the pictures do); this is important: if you impose planar structures you lose the symmetries and get a model for non-symmetric operads.

Given a tree T, there is a free operad $\Omega(T)$ in **Set** generated by T. Its colors are the edges of the tree T. The operations are generated by the vertices (not counting the root). For each vertex with inputs b, c, d and output edge a, there is an operation $p \in \Omega(T)(b, c, d; a)$ etc. For an edge c which isn't a leaf but has a single vertex sitting on top of it, you have $r \in \Omega(T)(; c)$. Then freely generate an operad (composites, symmetries) on this data. Note to define the generators you've chosen a planar structure but the resulting operad is independent of this.

You could define a morphism $S \to T$ of trees to be a morphism $\Omega(S) \to \Omega(T)$ of operads. There is also a completely combinatorial definition, which we will elucidate via examples.

"Face inclusions" correspond to growing an edge or to appending a new tree onto part of an existing tree at a vertex. "Degeneracies" can collapse unary operations but can't do any other quotienting. You also have automorphisms. If some vertex has n inputs then Σ_n

[27]The unit is the terminal category (the point), not the terminal operad (the commutative operad).

[28]Gijs: Should be called Ψ because Δ looks like a triangle and Ψ looks like a tree.

acts on this unlabeled set (even without labels; even without a planar representation) and so you get these maps.

Let $i\colon \mathbb{\Delta} \hookrightarrow \Omega$ denote the inclusion of $[n]$ as the tree with $n+1$ edges labeled $[n]$ and n vertices drawn so you only have unary operations.

Definition 8.5. $\mathbf{dSet} = \mathbf{Set}^{\Omega^{\mathrm{op}}}$

Denote the representables by $\Omega[T]$.

By left and right Kan extension you get

$$\begin{tikzcd} \mathbf{sSet} \arrow[r, bend left, "i_!"] \arrow[r, bend right, "i_*"'] & \mathbf{dSet} \arrow[l, "i^*" description] \end{tikzcd}$$

Note both $i_!$ and i_* are fully faithful.

Write $\eta = \Omega[|]$. Observe that any operad mapping to an operad with only unary operations must have only unary operations (must be a category). A similar observation produces an isomorphism $\mathbf{dSet}/\eta \cong \mathbf{sSet}$ that sits over $\mathbf{dSet}$ via the forgetful functor and $i_!$. (Aside i_* is weird; $i_!$ is the important one.)

In Ω, any morphism factors uniquely as a composite of degeneracies, followed by some isomorphisms, followed by face maps. This gives Ω the structure of a generalized Reedy category. In particular, this gives model structures.

Example 8.6. $\partial\Omega[T]$ is the union of all faces, or the colimit over all proper (non-iso) monos into T.

Example 8.7. $\Lambda^e T$, where e is an inner edge of T (an edge that is not a root and not a leaf), is the union of all faces except the face contracting e.

The assignment $T \mapsto \Omega(T)$ defines $N_d\colon \mathbf{Op}_{\mathbf{Set}} \to \mathbf{dSet}$ so that

$$\begin{tikzcd} \mathbf{Cat} \arrow[d, "N"'] & \mathbf{Op}_{\mathbf{Set}} \arrow[l, "j^*"'] \arrow[d, "N_d"] \\ \mathbf{sSet} & \mathbf{dSet} \arrow[l, "i^*"'] \end{tikzcd}$$

commutes. Like for categories, you can characterize the image of N_d by saying that these dendroidal sets have unique fillers for inner horns.

Homotopy theory of dendroidal sets.

Definition 8.8. The class of **normal monomorphisms** is the weak saturation of the class of boundary inclusions $\{\partial\Omega[T] \to \Omega[T]\}$.[29]

Lemma 8.9. *A monomorphism $f\colon X \to Y$ of dendroidal sets is a normal monomorphism if and only if for every three T, the automorphism group of T acts freely on the set $Y(T) - X(T)$.*

A dendroidal set X is **normal** if $\emptyset \to X$ is a normal monomorphism, i.e., if $\mathrm{Aut}(T)$ acts freely on $X(T)$ for each tree T. While it's hard to make a simplicial operad cofibrant, it's easy to normalize a dendroidal set: fix a normalization $E_\infty \xrightarrow{\sim}\!\!\!\!\rightarrow *$ (using say the small object argument) then take $X \times E_\infty$. Note that anything mapping to a cofibrant object is cofibrant (anything mapping equivariantly to something with a free action itself has a free action).

[29]These are exactly the Reedy monomorphisms.

Note Reedy category theory gives you a filtration of any dendroidal set as pushouts of normal monomorphisms (where the "skeletal filtrations" have to do with the number of vertices in the trees).

Definition 8.10. A map $X \to Y$ is an inner fibration if it has the right lifting property with respect to all inner horn inclusions $\Lambda^e[T] \hookrightarrow \Omega[T]$.

We'll call X a (dendroidal) ∞-operad if $X \to *$ is an inner fibration.

Theorem 8.11 (Cisinski-Moerdijk)**.** *Dendroidal sets has a left proper, combinatorial model structure such that the cofibrations are the normal monomorphisms and the fibrant objects are the ∞-operads. The induced model structure on* $\mathbf{dSet}/\eta$ *is the Joyal model structure.*

By an observation of Joyal, this completely characterizes the model structure. This will also be monoidal (once we define the tensor product). Note by the observation about cofibrants above, there is a Quillen equivalent model structure on $\mathbf{dSet}/E_\infty$ with all objects cofibrant.

Homotopy coherent nerve. We've already defined a nerve functor from operads in sets to dendroidal sets. What's interesting will be a homotopy coherent nerve from simplicial operads. We'll define this as the right adjoint to a left Kan extension of the thing you get by "fattening up" the images of the functor $T \mapsto \Omega(T)$. This is done by the Boardman-Vogt W construction, i.e., define

$$T \mapsto W(\Omega(T)) \colon \Omega \to \mathbf{sOp}.$$

The operads $\Omega(T)$ and $W(\Omega(T))$ will have the same colors. The idea is you replace the points in the spaces of operations by assigning a factor of Δ^1 to any subtree that has the given leaves and given root. So we get an adjunction

$$\begin{tikzcd} \mathbf{dSet} \arrow[r, shift left, "W_!"] \arrow[r, phantom, "\perp"] & \mathbf{sOp} \arrow[l, shift left, "W^*"] \end{tikzcd}$$

This extends the left adjoint to the ordinary homotopy coherent nerve.

$$\begin{tikzcd} \mathbf{sCat} \arrow[r, "j_!"] & \mathbf{sOp} \\ \mathbf{sSet} \arrow[u, "\mathfrak{C}"] \arrow[r, "i_!"'] & \mathbf{dSet} \arrow[u, "W_!"'] \end{tikzcd}$$

Taking right adjoints we get the desired commutative square above.

Theorem 8.12 (Cisinski-Moerdijk)**.** *$W_! \dashv W^*$ is a Quillen equivalence.*

Tensor products. Define $\Omega[S] \otimes \Omega[T] := N_d(\Omega(S) \otimes_{BV} \Omega(T))$. These are operads in sets so this is just the ordinary nerve. Then extend by colimits in each variable. This product extends the cartesian product of simplicial sets.

There is a combinatorial description of this in terms of "shuffles of trees." In analogy with the shuffles description of $\Delta^1 \times \Delta^1$ (two 2-simplices glued along an edge) the tensor of the tree with a single vertex, a single leaf, and a single root with itself can be described follows: color the vertex of one tree black and the other white. Then their tensor product is a quotient of the disjoint union of the two trees with two vertices (colored black on top then white on bottom in one tree and oppositely in the other), one inner edge, one leaf, and one root, quotiented by the relation that says if you contract the inner edges the result is the same (the half moon vertex with black on top and white on bottom is the same as the other half moon).

The Boardman-Vogt tensor product says if you have trees p and q (maybe only true if they have height 2), the tensor product is the quotient of the disjoint union of the tree with p grafted onto each leaf of q and the tree with q grafted onto each leave of p by the relation that says if you collapse all the inner edges the resulting things are the same. More generally, this tensor product is a quotient of the disjoint union of shuffles by the interchange relations. (Drew some pictures.)

Essential ingredients for the monoidal model structure:

Proposition 8.13. $\Lambda^e[S] \otimes \Omega[T] \cup \Omega[S] \otimes \partial\Omega[T] \hookrightarrow \Omega[S] \otimes \Omega[T]$ *is inner anodyne.*

This tensor product can be used to define internal homs, which then pullback to define an enrichment over simplicial sets. You can define weak equivalences to be things that, when homing into a fibrant object, induce weak equivalences between the hom-quasi-categories. Between fibrant objects there is a nicer description.

∞-operads (a la Lurie). Jacob's goal is to define operads in the language of category theory and then import it into quasi-categories. Let $\mathbb{F}$ be the category with objects the finite sets $\langle n\rangle$ with n-elements and whose maps are partial maps (only partially defined; compose where possible). There is an equivalence of categories between $\mathbb{F}$ and $\mathbf{Fin}_*$ (finite based sets) given by the functor that discards the basepoint.

Start with an operad O in sets and produce a functor $O^\otimes \to \mathbb{F}$ whose domain is the May-Thomason category of operators. Objects of $O^\otimes$ are tuples of colors of O. A map $f\colon (c_1, \ldots, c_m) \to (d_1, \ldots, d_n)$ is a map $\phi\colon \langle m\rangle \to \langle n\rangle$ in $\mathbb{F}$ together with operations $(c_j)_{j\in\phi^{-1}(i)} \to d_i$ of O for each $i \in \langle n\rangle$.

You can retrieve the operad O from this $\pi_O\colon O^\otimes \to \mathbb{F}$. To that end:

Definition 8.14. A map $\phi\colon \langle m\rangle \to \langle n\rangle$ in $\mathbb{F}$ is **inert** if for all $i \in \langle n\rangle$, $\phi^{-1}\{i\}$ is a singleton. E.g., there are inert morphisms $\rho^i\colon \langle n\rangle \to \langle 1\rangle$ that have $\{i\}$ as domain of definition.

(i) Given an inert $\phi\colon \langle m\rangle \to \langle n\rangle$ and an object $(c_1, \ldots, c_m)$, there is a canonical "projection" map in $O^\otimes$
$$(c_1, \ldots, c_m) \to (c_i)_{i\in\mathrm{dom}(\phi)}$$
that is π_O-coCartesian.

(ii) Now let ϕ be any map in $\mathbb{F}$ and consider the hom-sets of $O^\otimes$ sitting over ϕ. Then $O^\otimes((c_1, \ldots, c_m), (d_1, \ldots, d_n))_\phi \cong \prod_{i=1}^n O^\otimes((c_1, \ldots, c_m); d_i)_{\rho^i\circ\phi}$.

(iii) There are canonical isomorphisms of categories in the fibers over objects $O^\otimes_{\langle m\rangle} \cong (O^\otimes_{\langle 1\rangle})^{\times m}$.

Definition 8.15. An ∞-**operad** is a map of simplicial sets $O^\otimes \to N\mathbb{F}$ such that

(i) Given an inert $\phi\colon \langle m\rangle \to \langle n\rangle$ in $\mathbb{F}$ and any $C \in O^\otimes_{\langle m\rangle}$ there is a π_O-coCartesian lift of ϕ.

(ii) Can use the coCartesian lifts of the ρ^i to produce a map as in (ii) above which we require to be a homotopy equivalence.

(iii) The map $O^\otimes_{\langle m\rangle} \to (O^\otimes_{\langle 1\rangle})^{\times m}$ is a categorical equivalence.

Advantages of this definition is that we can use quasi-category theory. Disadvantages is there is a lot of redundancy in the definition.

Define $\mathbf{POp} = \mathbf{sSet}^+/N\mathbb{F}$ to be the category of ∞-preoperads. Write $N\mathbb{F}^\natural$ for $N\mathbb{F}$ with the inert edges marked. If $\pi\colon O^\otimes \to N\mathbb{F}$ is an ∞-operad, then an edge of $O^\otimes$ is said to be inert if its π-coCartesian. Write $O^{\otimes,\natural}$ for $O^\otimes$ with the inerts marked.

Theorem 8.16 (Lurie)**.** **POp** *carries a left proper, combinatorial, monoidal model structure so that the cofibrations are the monomorphisms and the fibrant objects are things of the form* $O^{\otimes,\natural}$.

Note because $\mathbb{F}$ is skeletal, it turns out the monoidal structure is only symmetric up to weak equivalence.

Q. What about algebras? In all cases, they're just defined to be morphisms of operads to the operad introduced above for a symmetric monoidal category.

$\boxed{\textbf{11/4}}$

9. Unicity I (Clark Barwick)

Theorem 9.1 (unicity)**.** *Let* $\mathbf{Thy}_{(\infty,n)}$ *denote the simplicial set of quasi-categories* C*, such that* C *is a model of the homotopy theory of* (∞, n)*-categories, with equivalences between them. Then* $\mathbf{Thy}_{(\infty,n)} \simeq B(\mathbb{Z}/2)^n$.

A corollary of this theorem is that the $(\infty, n+1)$-category of (∞, n)-categories is specified up to a contractible choice. We have yet to say what a "model of the homotopy theory of (∞, n)-categories" means. Before discussing the axioms, let's consider our favorite example.

Example 9.2. $\mathbf{CSS} \in \mathbf{Thy}_{(\infty,1)}$ is constructed by freely generated a homotopy theory from $\mathbb{\Delta}$: i.e., start with $\mathcal{P}(\mathbb{\Delta}) = \mathrm{Fun}(\mathbb{\Delta}^{\mathrm{op}}, \mathcal{S})$ and then impose relations. Recall the Yoneda embedding preserves few colimits, so we ask that

$$y[1] \cup_{y[0]} y[1] \cup_{y[0]} \cdots \cup_{y[0]} y[1] \to y[n]$$

is an equivalence. We have a 2-of-6 condition, asking that

$$y[3] \cup_{y[1] \cup y[1]} (y[0] \cup y[0]) \to y[0]$$

is an equivalence, where the two 1-simplex map the edges 02 and 13.

The upshot, expressed as a single universal property, is that colimit preserving functors $\mathbb{\Delta} \hookrightarrow \mathbf{CSS} \to C$ give rise to $\mathrm{Fun}^L(CSS, C) \hookrightarrow \mathrm{Fun}(\mathbb{\Delta}, C)$ which can be identified with the full subcategory spanned by those $h\colon \mathbb{\Delta} \to C$ satisfying (by analogy) the two displayed conditions.

This isn't how the axiomatization is going to go. We'd hope that $\mathrm{Aut}(\mathbf{CSS}) \cong \mathrm{Aut}(\mathbb{\Delta}) \cong \mathbb{Z}/2$. It is true that an automorphism of **CSS** is colimit preserving so is determined by its restriction to $\mathbb{\Delta}$, but the image of $\mathbb{\Delta}$ isn't necessarily fixed by the automorphism.

For (∞, n), we have at least two variants: $\mathbf{CSS}(\mathbb{\Delta}^n)$ and $\mathbf{CSS}(\Theta_n)$. Similar remarks apply to those cases.

Our first objective is to find a set of generators (in a sense made precise below) that are characterized by an ∞-categorical property so that it is invariant under taking equivalences.

$$\begin{tikzcd} C \arrow[r] & C \\ G \arrow[u, phantom, "\cup"] \arrow[r, dashed] & G \arrow[u, phantom, "\cup"] \end{tikzcd}$$

Example 9.3. $\mathcal{S} \in \mathbf{Thy}_{(\infty,0)}$ is defined by $\mathcal{P}(*)$. Of course the point, as a terminal object, is preserved under equivalences. But another way to think about this example is to start from finite sets and form $\mathcal{P}(\mathbf{FinSet})$. You get $\mathcal{S}$ by imposing the relation that $y(S \cup T) \simeq y(S) \cup y(T)$.

Definition 9.4. An object $X \in C$ is 0-**truncated** or simply **truncated** if for all $Y \in C$, $\mathrm{Map}_C(Y, X)$ is discrete.

Note this notion is preserved by equivalences, i.e., is "quasi-categorical." The previous example shows that the theory of $(\infty, 0)$-categories is generated by its 0-truncated objects. Discrete objects of **CSS** are ordinary categories with no non-trivial automorphisms (such as Δ^n for any n). Generalizing to higher dimensions, the role of the 0-truncated objects is played by so-called gaunt categories.

Definition 9.5 (term due to CS-P)**.** A (strict) n-category is **gaunt** if the only k-isomorphisms are identities.

We have factorizations

$$\begin{tikzcd} \mathbb{\Delta}^n \arrow[rr] \arrow[dr] & & \mathbf{CSS}(\mathbb{\Delta}^n) \\ & \mathbf{Gaunt}_n \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} \Theta_n \arrow[rr] \arrow[dr] & & \mathbf{CSS}(\Theta_n) \\ & \mathbf{Gaunt}_n \arrow[ur] & \end{tikzcd}$$

Observation 9.6. For $\mathbf{CSS}(\mathbb{\Delta}^n)$ and $\mathbf{CSS}(\Theta_n)$, the truncated objects are the gaunt guys. Equivalently, $\mathrm{Fun}(\mathbb{\Delta}^n, \mathbf{Set}) \to L_S \mathrm{Fun}(\mathbb{\Delta}^n, \mathbf{Set}) \simeq \mathbf{Gaunt}_n$ and similarly for Θ_n.

For $n = 2$, the localization collapses the vertical 1-simplices in $\mathbb{\Delta} \times \mathbb{\Delta}$ to equivalences, changing the squares into globes.

$$\begin{array}{ccccccc} \bullet & \to & \bullet & \to & \bullet & \to & \bullet \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \bullet & \to & \bullet & \to & \bullet & \to & \bullet \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \bullet & \to & \bullet & \to & \bullet & \to & \bullet \end{array}$$

So the gaunt categories are the same for both $\mathbb{\Delta}^n$ and Θ_n.

Write $\mathrm{Aut}(C) \subset \mathrm{Fun}(C, C)$ for the full subcategory spanned by the equivalences.

Proposition 9.7. $\mathrm{Aut}(\mathbf{Gaunt}_n) \simeq (\mathbb{Z}/2)^n$.

Corollary 9.8. *If* $\mathbf{Gaunt}_n$ *strongly generates* C *and if the truncated objects* $\tau_{\leq 0} C \simeq \mathbf{Gaunt}_n$, *then* $\mathrm{Aut}(C) \simeq (\mathbb{Z}/2)^n$.

Proof. We have inclusions

$$\begin{tikzcd} \mathrm{Aut}(C) \arrow[r] \arrow[d] & \mathrm{Fun}^L(C, C) \arrow[d] \\ \mathrm{Aut}(\mathbf{Gaunt}_n) \arrow[r] & \mathrm{Fun}(\mathbf{Gaunt}_n, C) \end{tikzcd}$$

which shows that something is fully faithful Essentially surjective then follows from restricting $C \to C$ to $\mathbf{Gaunt}_n \to \mathbf{Gaunt}_n$. □

Next objective is to get connectedness. It turns out this is kind of hard.

We're looking for some localization $\mathcal{P}(\mathbf{Gaunt}_n^\omega) \to L_S \mathcal{P}(\mathbf{Gaunt}_n^\omega)$, where the ω denotes a restriction to compact objects (covariant representables commuting with filtered colimits). The question is which homotopy colimits do not introduce new homotopy theory. The problem is we don't know how to characterize the S in this localization.

There's a second part of this problem: we need to find a subcategory of $\mathbf{Gaunt}_n$ that encapsulates the "good stuff" about (∞, n)-categories. To make this precise let's think about what we like about (∞, n)-categories.

(i) If X, Y are (∞, n)-categories, then $\mathrm{Fun}(X, Y) \in \mathbf{Cat}_{(\infty,n)}$.
(ii) In practice, we do more ...

Definition 9.9. A **distributor** is a **profunctor**, i.e., $B \nrightarrow A$ is $A^{\mathrm{op}} \times B \to \mathbf{Set}$.

Example 9.10. An adjunction is conveniently encoded as an isomorphism of represented distributors $A(f, 1) \cong B(1, u)$.

The category of distributors has an internal hom: given $X : A^{\mathrm{op}} \times B \to \mathbf{Set}$ and $Y : C^{\mathrm{op}} \times D \to \mathbf{Set}$ define $[X, Y] : \mathrm{Fun}(A, C)^{\mathrm{op}} \times \mathrm{Fun}(B, D) \to \mathbf{Set}$ by $(\phi, \psi) \mapsto \mathrm{Nat}(X, Y \circ (\phi, \psi))$.

We want to get the same feature on (∞, n)-categories. A **correspondence** is a functor $X \to \mathbb{2}$. From the (categorical) Grothendieck construction, $B \xrightarrow{F} A$ can be encoded as a correspondence (in two different ways).

Given a correspondence $X \to \mathbb{2}$, define $A = X_0$ and $B = X_1$. There is a natural distributor $X : A^{\mathrm{op}} \times B \to \mathbf{Set}$ defined in the obvious way. This defines an equivalence of categories between **Dist** and the subcategory of $\mathbf{Cat}/\mathbb{2}$ of two-sided conduche fibrations or something like that, which then of course preserves limits, e.g., products. The product in $\mathbf{Cat}/\mathbb{2}$ (pullback over $\mathbb{2}$) has a right adjoint, and in particular preserves colimits

$$X \times_{\mathbb{2}} (\operatorname*{colim}_{\alpha} Y_\alpha) \cong \operatorname*{colim}_{\alpha}(X \times_{\mathbb{2}} Y_\alpha).$$

This has to do with an axiom from topos theory: colimits are universal, i.e., colimits are preserved by base change. Note this isn't true for all slice categories in **Cat**. But it's okay to replace $\mathbb{2}$ by $\mathbb{1}$ (in which case we just have ordinary objects).

Definition 9.11. Define $\Upsilon_n \subset \mathbf{Gaunt}_n$ to be the smallest full subcategory that

- contains all the generic cells in each dimension $C_0 = \bullet$, $C_1 = \bullet \to \bullet$, C_2 the walking 2-cell, ..., C_n the walking n-cell
- is closed under retracts
- is closed under $- \times_{C_k} -$.

Q (Jeremy Hahn)**.** What's the motivation for the retract axiom? **A.** It's something technical: we didn't want to fiddle with the ambiguity of the idempotent completion in our generators. (Sets of free generators of a homotopy theory can differ, but their idempotent completions are all the same.)

For example, on account of the pullback diagram

$$\begin{tikzcd} \partial C_{k-1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C_{k-1} \arrow[d] \\ C_{k-1} \arrow[r] & C_k \end{tikzcd}$$

you have boundaries (at least if $k \leq n-2$). For $n = 1$, pullback over C_0 means you have products, say C_1^n. Now retract closure gives you the categories that are the objects of $\mathbb{\Delta}$. Note this is big: $\Theta_n \subset \Upsilon_n$.

Now let's return to the opening theorem.

Theorem 9.12 (unicity)**.** *Let* $\mathbf{Thy}_{(\infty,n)}$ *denote the simplicial set of (presentable) quasi-categories C, with equivalences between them, such that there exists a fully faithful functor* $f : \Upsilon_n \to \tau_{\leq 0} C \to C$ *so that*

(i) f ***strongly generates*** C.
(ii) For any map $X \to C_k$, the endofunctor $X \times_{C_k} -$ of C/C_k preserves colimits.
(iii) Fundamental pushouts are preserved:

- *$f(c_i) \cup_{f(\partial c_i)} f(c_i) \xrightarrow{\sim} f(\partial c_{i+1})$ (including the degenerate case $\emptyset \xrightarrow{\sim} f(\emptyset)$).*
- *$f(c_j) \cup_{f(c_i)} f(c_j) \xrightarrow{\sim} f(c_j \cup_{c_i} c_j)$ if $i < j$.*
- *$f(c_{i+j} \cup_{f(c_j)} f(c_{j+k}) \cup_{f(\sigma^{i+1}(c_{j-1} \times c_{k-1}))} f(c_{i+k}) \cup_{f(c_i)} f(c_{i+j}) \xrightarrow{\sim} f(c_{i+j} \times_{c_i} c_{i+k})$*
- *$(f(\sigma^k[3]) \cup_{f(\sigma^k[1]) \cup f(\sigma^k[1])} (f(\sigma^k[0]) \cup f(\sigma^k[0])]) \simeq f(c_k)$*

(iv) *If $(\mathcal{D}, g)$ is a presentable quasi-category strongly generated by $g \colon \Upsilon_n \to \mathcal{D}$ satisfying these conditions, then there exists a localization*

$$\begin{tikzcd} \Upsilon_n \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "\cong"] & \mathcal{D} \\ C \arrow[ur, "F"'] & \end{tikzcd}$$

Then $\mathbf{Thy}_{(\infty,n)} \simeq B(\mathbb{Z}/2)^n$.

The notation $\sigma(c)$ is the gaunt n-category with two objects x, y and $\hom(x, y) = c$.

The third axiom has to do with the 2-of-6 localization for CSS, without which you get a model for the wrong homotopy theory (with weak equivalences just the isomorphisms). In $\mathbf{Gaunt}_n$ we didn't know which colimits should not produce any new homotopy theory, but in Υ_n we do, which was the point of making the restriction.

Let's outline the proof.

(i) Show that $\mathbf{Thy}_{(\infty,n)}$ is inhabited.
(ii) If C satisfies (1)-(4), then $\tau_{\leq 0} \simeq \mathbf{Gaunt}_n$.
(iii) Show that the usual suspects are indeed vertices of $\mathbf{Thy}_{(\infty,n)}$.

Q. How does this compare with Toen? **A.** His axioms were really built around the CSS example. Here this is less the case. In particular, axiom (2) has no analog in the $n = 1$ case.

10. Unicity II (Clark Barcwick)

Recall $\mathbf{Gaunt}_n$ in the full subcategory of strict n-categories so that the only isomorphisms at any level are identities. Then $\Upsilon_n \subset \mathbf{Gaunt}_n$ is the smallest full subcategory such that

(i) $C_k \in \Upsilon_n$,
(ii) Υ_n is closed under retracts.
(iii) Υ_n is closed under $- \times_{C_k} -$ (fiber products over cells).

Left overs.

(i) $\partial C_k \in \Upsilon_n$ only if $0 \leq k \leq n-1$.
(ii) Analyze what does $C_i \times_{C_j} C_k$ look like, i.e., what is a decomposition as a colimit of cells?

This second thing will help us motivate some of the "fundamental pushouts" introduced last time. Consider

$$\begin{tikzcd} & C_k \arrow[d, "\psi"] \\ C_i \arrow[r, "\phi"'] & C_j \end{tikzcd}$$

Two options for ϕ:

(i) $\phi = \sigma(C_{i-1} \to C_{n-1})$
(ii) $C_i \to C_0 \to C_j$.

That is, either ϕ is a suspension of something that happens in lower dimensions, or it is constant. Observe that suspension preserves fiber products (though not all limits), so we're good in that case. So consider

$$\begin{tikzcd} F \times C_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C_k \arrow[d, "\psi"] \\ C_i \arrow[r] & C_0 \arrow[r] & C_j \end{tikzcd}$$

Again we have two cases: either $F =$ or $F = C_m$, in which case $C_i \times_{C_j} C_k = C_i \times C_m$. So we've reduced to understanding products of cells.

Example 10.1. $C_1 \times C_1 = (C_1 \cup_{C_0} C_1) \cup_{C_1} (C_1 \cup_{C_0} C_1)$. Note these colimits are in **Cat** so, e.g., each $C_1 \cup_{C_0} C_1$ includes both the composable arrows and their composite.

Example 10.2. $C_2 \times C_1 = (C_1 \cup_{C_0} C_2) \cup_{C_2} (C_2 \cup_{C_0} C_1)$. Here the C_2 we're gluing along is properly thought of as $\sigma(C_1 \times C_0)$.

More generally

$$C_i \times C_m = (C_i \cup_{C_0} C_m) \cup_{\sigma(C_{i-1} \times C_{m-1})} (C_m \cup_{C_0} C_i).$$

Review of the axioms. Given a presentable quasi-category $\mathcal{C}$ and a functor $f\colon \Upsilon_n \to \tau_{\leq 0}\mathcal{C}$:

(c.1) strong generator: f strongly generates $\mathcal{C}$, meaning $\mathcal{C}$ is a localization of a presheaf category on Υ_n

(c.2) correspondences have internal homs: for all $\eta\colon X \to f(C_i)$,

$$- \times_{f(C_i)} X = \eta^* \colon \mathcal{C}/f(C_i) \to \mathcal{C}/X$$

preserves colimits

(c.3) fundamental pushouts:

- (i) $f(c_i) \cup_{f(\partial c_i)} f(c_i) \xrightarrow{\sim} f(\partial c_{i+1})$ (including $\emptyset \simeq f(\emptyset)$).
- (ii) $f(c_j) \cup_{f(c_i)} f(c_j) \xrightarrow{\sim} f(c_j \cup_{c_i} c_j)$ if $i < j$.
- (iii) $f(c_{i+j} \cup_{f(c_j)} f(c_{j+k}) \cup_{f(\sigma^{i+1}(c_{j-1} \times c_{k-1}))} f(c_{i+k}) \cup_{f(c_i)} f(c_{i+j}) \xrightarrow{\sim} f(c_{i+j} \times_{c_i} c_{i+k})$
- (iv) $(f(\sigma^k[3]) \cup_{f(\sigma^k[1]) \cup f(\sigma^k[1])} (f(\sigma^k[0]) \cup f(\sigma^k[0])]) \simeq f(c_k)$.

(c.4) versality: If you've got an $(\mathcal{D}, g)$ satisfying (c.1-3) then there exists a localization

$$\begin{tikzcd} \mathcal{C} \arrow[rr] & & \mathcal{D} \\ & \Upsilon_n \arrow[ul, "f"] \arrow[ur, "g"'] & \end{tikzcd}$$

Goals.

- (i) Produce an example
- (ii) Show that $\tau_{\leq 0}\mathcal{C} \simeq \mathbf{Gaunt}_n$.

From the dense inclusion $\Upsilon_n \hookrightarrow \mathbf{Gaunt}_n$ we have a fully faithful functor $\nu\colon \mathbf{Gaunt}_n \to \mathbf{Set}^{\Upsilon_n^{op}}$. $\mathbf{Gaunt_n}$ is local with respect to the maps (i)-(iv) above, taking f to be the Yoneda embedding. E.g., consider

$$y(c_i) \cup_{y(\partial c_i)} y(c_i) \to y(\partial c_{i+1}).$$

Then if H is gaunt, the map

$$\hom(y(\partial c_{i+1}), H) \to \hom(y(c_i) \cup_{y(\partial c_i)} y(c_i), H)$$

is a bijection. Note this condition for the last “2-of-6” map is encoding gauntness: it tells us that any isomorphism is an identity. The localness with respect to the other three classes is also true for any strict n-category.

Write $S_{0,0}$ for the union of (i), (ii), (iii), and (iv). Let S_0 then be the smallest class that is closed under isomorphisms, under $X \times_{C_i} -$ for $X \in \mathbf{Gaunt}_n$. Let S be the saturated class generated by S_0.

Definition 10.3. Let $\mathbf{Cat}_{(\infty,n)} = S^{-1}\mathcal{P}(\Upsilon_n)$, the localization at presheaves of spaces.

Observe

- this clearly satisfies (c.1) and (c.3)
- it also satisfies (c.2)
- (c.4) follows from the universal property of localizations.

This completes our first goal. For the second, nerves of gaunt n-categories are easily seen to be local with respect to S_0. But $X \in \tau_{\leq 0}\mathbf{Cat}_{(\infty,n)} = S^{-1}\mathbf{Set}^{\Upsilon_n^{\mathrm{op}}}$. We want to find compositions on $X|_{\mathbb{G}_n}\mathbb{G}_n^{\mathrm{op}} \to \mathbf{Set}$, writing $\mathbb{G}_n$ for the n-truncation of the category of globular sets. From $C_j \to C_j \cup_{C_i} C_j$ we get

$$X(C_j) \times_{X(C_i)} X(C_j) \cong X(C_j \cup_{C_i} C_j) \to X(C_j)$$

which gives X the structure of an n-category.

Corollary 10.4. $\mathbf{Thy}_{(\infty,n)} \simeq (B\mathbb{Z}/2)^n$.

Now how do we find examples? We’ll see there’s a different way to check the axioms in the case that your model comes packaged as generators and relations.

Checking the axioms. We’ll think about the examples $\mathbf{CSS}(\mathbb{\Delta}^n)$ and $\mathbf{CSS}(\Theta_n)$.

Observe that the generators are ordinary categories, not $(\infty,1)$-categories, that we’ll commonly denote by $\mathcal{R}$. Suppose the relations are given by a class T_0 whose saturation $\overline{T_0} = T$. In addition, we’ll need a functor $i\colon \mathcal{R} \to \Upsilon_n$, that need not be fully faithful. From this data, we get a string of adjoints

$$\begin{tikzcd} \mathcal{P}(\mathcal{R}) \arrow[r, bend left, "i_!"] \arrow[r, phantom, shift left=1.5, "\perp"] \arrow[r, phantom, shift right=1.5, "\perp"] \arrow[r, bend right, "i_*"'] & \mathcal{P}(\Upsilon_n) \arrow[l, "i^*"'] \end{tikzcd}$$

Theorem 10.5. *Suppose*

(r.1) $i^*(S_0) \subset T$
(r.2) $i_!(T_0) \subset S$
(r.3) for all $r \in \mathcal{R}$, $r \to i^*i(r) \in T$
(r.4) i hits all the cells

Then i^* *makes* $T^{-1}\mathcal{P}(\mathcal{R}) \simeq \mathbf{Cat}_{(\infty,n)}$.

The proof will depend upon a lemma. We have $\mathbf{Cat}_{(\infty,n)} = S^{-1}\mathcal{P}(\Upsilon_n)$.

Lemma 10.6. *Cells detect equivalences: if* $X \to Y$ *is a map and if* $\mathrm{Map}(C_k, X) \xrightarrow{\sim} \mathrm{Map}(C_k, Y)$ *for all k then* $X \xrightarrow{\sim} Y$.

Proof. We have

$$\mathbb{G}_n = \Upsilon_n^{(0)} \subset \Upsilon_n^{(1)} \subset \cdots \subset \Upsilon_n$$

where $\Upsilon_n^{(k)} = \{X \in \Upsilon_n \mid X$ is the colimit of a diagram $K^{\triangleright} \xrightarrow{P} \mathbf{Cat}_{(\infty,n)}$ so that $P(K) \subset \mathbf{Gaunt}_n^{(k-1)}\}$. What is meant here is that all the *maps* in the diagrams also have to be in the subcategories (this is why taking colimits of colimits gives us something new). So to start, we're only allowed basic maps between the cells because this is all that's in $\mathbb{G}_n$.

What we need to show is that $\Upsilon_n = \cup_{k\geq 0}\Upsilon_n^{(k)} =: U$. By definition Υ_n is the smallest subcategory containing $\mathbb{G}_n$ and closed under retracts and $-\times_{C_k}-$. U is closed under retracts because retracts are a particular kind of colimit.

Consider the assertion

$(*)_k$ If $X, Y \in \Upsilon_n^{(k)}$, then for all $X \to C_i$, $Y \to C_i$, $X \times_{C_i} Y \in \Upsilon_n^{(k+n+1)}$.

For $k = 0$ this is just true. Then $(*)_k \Rightarrow (*)_{k+1}$ follows from (c.2). □

This lemma wasn't just a technicality. It's really essential.

Proof of theorem. Right adjoints carry local objects to local objects, which means we'll consider functors

$$\begin{tikzcd} T^{-1}\mathcal{P}(\mathcal{R}) \arrow[r, bend left, "i_!"] & S^{-1}\mathcal{P}(\Upsilon_n) = \mathbf{Cat}_{(\infty,n)} \arrow[l, "i^*"'] \arrow[l, bend left, "i_*"] \end{tikzcd}$$

Now (r.3) implies that for each generator $r \in \mathcal{R}$, $r \xrightarrow{\sim} i^* L_S i_! r \in T^{-1}\mathcal{P}(\mathcal{R})$ which implies that $\mathrm{id} \to i^* L_S i_!$ is an equivalence.

Now consider $\mathrm{id} \to i_* i^*$. For all X we claim that $\mathrm{Map}(C_k, X) \xrightarrow{\sim} \mathrm{Map}(C_k, i_* i^* X) \simeq \mathrm{Map}(ir_k i_* i^* X)$, by (r.4), which is equivalent to $\mathrm{Map}(i^*(ir_k), i^* X)$. This is equivalent, by (r.3), to $\mathrm{Map}(r_k, i^* X) \simeq \mathrm{Map}(i_! r_k, X) \simeq \mathrm{Map}(ir_k, X) \simeq \mathrm{Map}(c_k, X)$. □

When we wrote this paper, we didn't know that complete Segal spaces on $\mathbb{\Delta}^n$ and complete Segal spaces on Θ_n were going to be equivalent. We used this result to prove it. Some general observations.

(i) (r.1) is the hard one to check
(ii) This condition can be decomposed as
  (r.1a) $i^*(S_{0,0}) \subset T$
  (r.1b) for all $U \to V \in T$, $V \to i^* C_k$, and $H \to C_k \in \Upsilon_n$ form

$$\begin{tikzcd} U' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & V' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & i^* H \arrow[d] \\ U \arrow[r] & V \arrow[r] & i^* C_k \end{tikzcd}$$

  and check that $U' \to V'$ is in T.
(iii) (r.3) is obvious if i happens to be fully faithful (e.g. for Θ_n).

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