The formal theory of adjunctions, monads, algebras, and descent

Emily Riehl

Harvard University
http://www.math.harvard.edu/~eriehl

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Formal theory of adjunctions, monads, algebras, descent

Joint with Dominic Verity.

Plan

Part I. adjunctions and monads

- **context:**
$$\begin{tikzcd} 2\text{-CAT} \arrow[r, hook, "\perp", "\mathrm{incl}"'] & (\infty,2)\text{-CAT} \arrow[l, bend right, "\mathrm{htpy}"'] \end{tikzcd}$$
- **Theorem.** Any adjunction in a homotopy 2-category extends to a homotopy coherent adjunction in the $(\infty, 2)$-category.

(Interlude on weighted limits.)

Part II. algebras and descent

- **definitions** of algebras, descent objects: via weighted limits
- **proofs** of monadicity, descent theorems: all in the weights!

Some basic shapes

categories with a monad	weight ("shape") for
$\Delta_+ = \bullet \rightarrow \bullet \substack{\rightarrow \\ \leftarrow \\ \rightarrow} \bullet \cdots$	underlying object
$\Delta = \bullet \substack{\rightarrow \\ \leftarrow \\ \rightarrow} \bullet \substack{\rightarrow \\ \leftarrow \\ \rightarrow \\ \leftarrow \\ \rightarrow} \bullet \cdots$	descent object
$\Delta_\infty = \bullet \substack{\rightarrow \\ \leftarrow} \bullet \substack{\rightarrow \\ \leftarrow \\ \rightarrow \\ \leftarrow} \bullet \cdots$	object of algebras

Moreover:

$$\begin{tikzcd} \Delta_+ \arrow[dr, dashed, bend right] \arrow[dr, phantom, "\dashv"] & & \Delta \arrow[ll, hook'] \arrow[dl, dashed, bend left] \arrow[dl, phantom, "\vdash"] \\ & \Delta_\infty \arrow[ul, hook] \arrow[ur, hook'] & \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} B \arrow[rr] \arrow[dr] \arrow[dr, phantom, "\perp"] & & \mathrm{dsc}B \arrow[dl] \arrow[dl, phantom, "\perp"] \\ & \mathrm{alg}B \arrow[ul, dashed, bend left] \arrow[ur, dashed, bend right] & \end{tikzcd}$$

$(\infty, 2)$-categories

An $(\infty, 2)$-**category** is a simplicially enriched category whose hom-spaces are quasi-categories.

$$\begin{tikzcd} \mathrm{Cat} \arrow[r, hook, "N"'] \arrow[r, phantom, "\perp"] & \mathrm{qCat} \arrow[l, bend right, "\mathrm{ho}"'] \end{tikzcd} \qquad \rightsquigarrow \qquad \begin{tikzcd} 2\text{-}\mathrm{CAT} \arrow[r, hook, "\mathrm{incl}"'] \arrow[r, phantom, "\perp"] & (\infty, 2)\text{-}\mathrm{CAT} \arrow[l, bend right, "\mathrm{htpy}\ 2\text{-}\mathrm{cat}"'] \end{tikzcd}$$

Examples.

- 2-categories: categories, monoidal categories, accessible categories, algebras for any 2-monad, . . .
- $(\infty, 2)$-categories: quasi-categories, complete Segal spaces, Rezk objects, . . .

The free adjunction

$\mathbf{Adj}$:= the **free adjunction**, a 2-category with

- objects $+$ and $-$
- $\hom(+,+) = \hom(-,-)^{\mathrm{op}} := \mathbb{\Delta}_+$
- $\hom(-,+) = \hom(+,-)^{\mathrm{op}} := \mathbb{\Delta}_\infty$

Theorem (Schanuel-Street). Adjunctions in a 2-category $\mathbf{K}$ correspond to 2-functors $\mathbf{Adj} \to \mathbf{K}$.

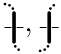
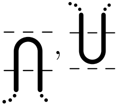
$$\begin{tikzcd}[column sep=huge]
\mathrm{id} \arrow[r, "\eta" description] & uf \arrow[r, shift left=6, "\eta" description] \arrow[r, leftarrow, "u\epsilon" description] \arrow[r, shift right=6, "uf\eta" description] & ufuf \arrow[r, shift left=12, "\eta" description] \arrow[r, shift left=6, leftarrow, "u\epsilon" description] \arrow[r, "uf\eta" description] \arrow[r, shift right=6, leftarrow, "ufu\epsilon" description] \arrow[r, shift right=12, "ufuf\eta" description] & ufufuf \cdots
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=huge]
u \arrow[r, shift left=3, "\eta" description] \arrow[r, shift right=3, leftarrow, "u\epsilon" description] & ufu \arrow[r, shift left=9, "\eta" description] \arrow[r, shift left=3, leftarrow, "u\epsilon" description] \arrow[r, shift right=3, "uf\eta" description] \arrow[r, shift right=9, leftarrow, "ufu\epsilon" description] & ufufu \arrow[r, shift left=15, "\eta" description] \arrow[r, shift left=9, leftarrow, "u\epsilon" description] \arrow[r, shift left=3, "uf\eta" description] \arrow[r, shift right=3, leftarrow, "ufu\epsilon" description] \arrow[r, shift right=9, "ufuf\eta" description] \arrow[r, shift right=15, leftarrow, "ufufu\epsilon" description] & ufufufu \cdots
\end{tikzcd}$$

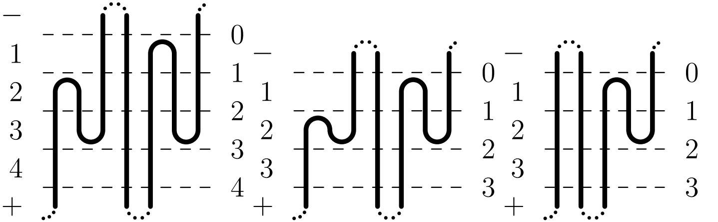
The free homotopy coherent adjunction

Theorem (Schanuel-Street). Adjunctions in a 2-category $\mathbf{K}$ correspond to 2-functors $\mathbf{Adj} \to \mathbf{K}$.

A **homotopy coherent adjunction** in an $(\infty,2)$-category $\mathbf{K}$ is a simplicial functor $\mathbf{Adj} \to \mathbf{K}$.

data in $\mathbf{Adj}$: $-, +$;  ;  ; ...

n-arrows are **strictly undulating squiggles** on $n+1$ lines



Homotopy coherent adjunctions

A **homotopy coherent adjunction** in an $(\infty, 2)$-category $\mathbf{K}$ is a simplicial functor $\mathbf{Adj} \to \mathbf{K}$.

Theorem. Any adjunction in the homotopy 2-category of an $(\infty, 2)$-category extends to a homotopy coherent adjunction.

Theorem. Moreover, the spaces of extensions are contractible Kan complexes.

Upshot: there is a good supply of adjunctions in $(\infty, 2)$-categories.

Proposition. $\mathbf{Adj}$ is a simplicial computad (*cellularly* cofibrant).

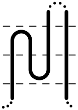
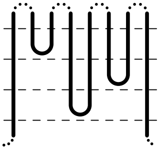
Homotopy coherent monads

$\mathbf{Mnd}$:= full subcategory of $\mathbf{Adj}$ on $+$.

A **homotopy coherent monad** in an $(\infty, 2)$-category $\mathbf{K}$ is a simplicial functor $\mathbf{Mnd} \to \mathbf{K}$., i.e.,

- $+ \mapsto B \in \mathbf{K}$
- $\mathbb{\Delta}_+ \to \hom(B, B) =:$ the **monad resolution**

$$\begin{tikzcd} \mathrm{id}_B \arrow[r, "\eta"] & t \arrow[r, shift left=3, "\eta"] \arrow[r, leftarrow, "\mu"] \arrow[r, shift right=3, "t\eta"'] & t^2 \arrow[r, shift left=6, "\eta"] \arrow[r, leftarrow, shift left=3, "\mu"] \arrow[r, "t\eta"] \arrow[r, leftarrow, shift right=3, "t\mu"'] \arrow[r, shift right=6, "tt\eta"'] & t^3 \cdots \end{tikzcd}$$

and higher data: $$\begin{tikzcd} & t^2 \arrow[dr, "\mu"] & \\ t \arrow[ur, "\eta t"] \arrow[rr, equal] \arrow[rr, phantom, shift left=3, "\sim"] & & t \end{tikzcd} \quad \leftrightsquigarrow$$  

Warning: A monad in the homotopy 2-category need not lift to a homotopy coherent monad.

Interlude on weighted limits

Let $\mathbf{K}$ be a 2-category or an $(\infty, 2)$-category, $\mathbf{A}$ a small 2-category.

$$\begin{array}{ccccc} \text{weight}^{\text{op}} & \times & \text{diagram} & \mapsto & \text{limit} \\ \pitchfork & & \pitchfork & & \pitchfork \\ (\mathbf{Cat}^{\mathbf{A}})^{\text{op}} & \times & \mathbf{K}^{\mathbf{A}} & \xrightarrow{\{-,-\}_{\mathbf{A}}} & \mathbf{K} \end{array}$$

Facts:

- $\{W, D\}_{\mathbf{A}} :=$ some limit formula using cotensors
- a representable weight evaluates at the representing object
- colimits of weights give rise to limits of weighted limits

Cellular weighted limits

Example ($\mathbf{A} := b \xrightarrow{f} a \xleftarrow{g} c$). Define $W \in \mathbf{sSet}^{\mathbf{A}}$ by:

$$
\begin{tikzcd}
\hom_b \times \partial\Delta^0 \sqcup \hom_c \times \partial\Delta^0 \arrow[rr] \arrow[d] & & \emptyset \arrow[d] & \\
\hom_b \times \Delta^0 \sqcup \hom_c \times \Delta^0 \arrow[rr] & & \hom_b \sqcup \hom_c \arrow[d] \arrow[ul, phantom, "\ulcorner", very near start] & \hom_a \times \partial\Delta^1 \arrow[l, "f \sqcup g"'] \arrow[d] \\
\{W, -\}_{\mathbf{A}} =: \text{ comma object} & & W \arrow[ur, phantom, "\urcorner", very near start] & \hom_a \times \Delta^1 \arrow[l]
\end{tikzcd}
$$

A **cellular weight** is a cell complex in the projective model structure on $\mathbf{Cat}^{\mathbf{A}}$ or $\mathbf{sSet}^{\mathbf{A}}$.

Example: Bousfield-Kan homotopy limits.

Completeness hypothesis: $\mathbf{K}$ admits cellular weighted limits.

Algebras for a homotopy coherent monad

Fix a homotopy coherent monad: $\begin{array}{ccc} \mathbf{Mnd} & \to & \mathbf{K} \\ + & \mapsto & B \end{array}$

Goal: define the **object of algebras** $\mathrm{alg}B \in \mathbf{K}$ and the **monadic homotopy coherent adjunction** $\mathrm{alg}B \underset{u}{\overset{f}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} B$

$$\begin{tikzcd} & \mathbf{Adj} \arrow[dr, dashed] & \\ \mathbf{Mnd} \arrow[ur, hook] \arrow[rr] & & \mathbf{K} \\ + & \mapsto & B \end{tikzcd}$$

$B \in \mathbf{K}^{\mathbf{Mnd}}$ $\qquad$ $\mathbb{\Delta}_\infty \in \mathbf{Cat}^{\mathbf{Mnd}}$

$$\mathrm{alg}B := \{\mathbb{\Delta}_\infty, B\}_{\mathbf{Mnd}} = \mathrm{eq}\left(B^{\mathbb{\Delta}_\infty} \rightrightarrows B^{\mathbb{\Delta}_+ \times \mathbb{\Delta}_\infty}\right)$$

Algebras, continued

$$\mathrm{alg}B := \{\mathbb{\Delta}_\infty, B\}_{\mathbf{Mnd}} = \mathrm{eq}\left(B^{\mathbb{\Delta}_\infty} \rightrightarrows B^{\mathbb{\Delta}_+ \times \mathbb{\Delta}_\infty}\right)$$

Example: $\mathbf{K} = \mathbf{qCat}$. A vertex in $\mathrm{alg}B$ is a map $\mathbb{\Delta}_\infty \to B$ of the form:

$$\begin{tikzcd} b \arrow[r, shift left=1, "\eta"] & tb \arrow[l, shift left=1, "\beta"] \arrow[r, shift left=3, "\eta"] \arrow[r, shift right=1, "t\eta"] & t^2b \arrow[l, shift right=1, "\mu"'] \arrow[l, shift left=3, "t\beta"] \arrow[r, shift left=5, "\eta"] \arrow[r, shift left=1, "t\eta"] \arrow[r, shift right=3, "tt\eta"] & t^3b \cdots \arrow[l, shift right=3, "\mu"'] \arrow[l, shift left=1, "t\mu"] \arrow[l, shift left=5, "tt\beta"] \end{tikzcd}$$

and higher data, e.g.,

$$\begin{tikzcd} & tb \arrow[dr, "\beta"] & \\ b \arrow[ur, "\eta"] \arrow[rr, equal] \arrow[rr, phantom, shift left=3, "\sim"] & & b \end{tikzcd}$$

The monadic homotopy coherent adjunction

... is all in the weights!

$$
\begin{tikzcd}
\mathbf{Adj}^{\mathrm{op}} \arrow[r, "\text{yoneda}"] & \mathbf{Cat}^{\mathbf{Adj}} \arrow[r, "\text{res}"] & \mathbf{Cat}^{\mathbf{Mnd}} \arrow[r, "\{-,B\}_{\mathbf{Mnd}}"] & \mathbf{K}^{\mathrm{op}} \\
- \arrow[d, bend left, "u"] \arrow[r, phantom, "\mapsto"] & \hom_- \arrow[d, bend right] \arrow[r, phantom, "\mapsto"] & \mathbb{\Delta}_\infty \arrow[d, bend right] \arrow[r, phantom, "\mapsto"] & \mathrm{alg}B \arrow[d, bend left, "u"] \\
+ \arrow[u, bend left, "f"] \arrow[u, phantom, "\dashv"] \arrow[r, phantom, "\mapsto"] & \hom_+ \arrow[u, bend right] \arrow[u, phantom, "\vdash"] \arrow[r, phantom, "\mapsto"] & \mathbb{\Delta}_+ \arrow[u, bend right] \arrow[u, phantom, "\vdash"] \arrow[r, phantom, "\mapsto"] & B \arrow[u, bend left, "f"] \arrow[u, phantom, "\dashv"]
\end{tikzcd}
$$

Note: $\mathbf{Adj}$ a simplicial computad $\rightsquigarrow$ these weights are cellular.

Q: Doesn't this imply that up-to-homotopy monads have monadic adjunctions and hence are homotopy coherent?
A: No! Homotopy 2-categories don't admit cellular weighted limits.

Descent data for a homotopy coherent monad

Fix $B \in \mathbf{K}^{\mathbf{Mnd}}$. A **descent datum** is a coalgebra for the induced monad on the object of algebras.

$$\mathrm{dsc}B := \mathrm{coalg}(\mathrm{alg}(B)) \qquad\qquad \mathbb{\Delta} \in \mathbf{Cat}^{\mathbf{Mnd}}$$

$$\mathrm{dsc}B := \{\mathbb{\Delta}, B\}_{\mathbf{Mnd}} = \mathrm{eq}\left(B^{\mathbb{\Delta}} \rightrightarrows B^{\mathbb{\Delta}_+ \times \mathbb{\Delta}}\right)$$

Example ($\mathbf{K} = \mathbf{qCat}$). A vertex in $\mathrm{dsc}B$ is a map $\mathbb{\Delta} \to B$:

$$\begin{tikzcd} b \arrow[r, shift left=2, "\eta"] \arrow[r, shift right=2, "\gamma"'] & tb \arrow[l, "\beta" description] \arrow[r, shift left=4, "\eta"] \arrow[r, "t\eta" description] \arrow[r, shift right=4, "t\gamma"'] & t^2 b \cdots \arrow[l, shift right=2, "\mu"'] \arrow[l, shift left=2, "t\beta"] \end{tikzcd} \qquad \text{and higher data}$$

$$\begin{tikzcd} \mathbb{\Delta}_+ \arrow[dr, dashed] & & \mathbb{\Delta} \arrow[ll, hook'] \arrow[dl, dashed] \\ & \mathbb{\Delta}_\infty \arrow[ul, hook] \arrow[ul, phantom, "\perp"] \arrow[ur, hook] \arrow[ur, phantom, "\perp"] & \end{tikzcd} \text{ in } \mathbf{Cat}^{\mathbf{Mnd}} \quad \rightsquigarrow \quad \text{in } \mathbf{K} \begin{tikzcd} B \arrow[rr] \arrow[dr] & & \mathrm{dsc}B \arrow[dl] \\ & \mathrm{alg}B \arrow[ul, dashed] \arrow[ul, phantom, "\perp"] \arrow[ur, dashed] \arrow[ur, phantom, "\perp"] & \end{tikzcd}$$

Totalizations of cosimplicial objects in an object of $\mathbf{K}$

The monadicity and descent theorems require geometric realization of simplicial objects valued in an object of an $(\infty, 2)$-category.

An object $B \in \mathbf{K}$ **admits totalizations** iff there is an **absolute right lifting diagram** in $\mathrm{ho}\mathbf{K}$:

$$\begin{tikzcd} & B \arrow[d, "\mathrm{const}"] \\ B^{\blacktriangle} \arrow[ur, "\mathrm{tot}"] \arrow[r, "\mathrm{id}"'] \arrow[ur, phantom, shift right=3, "\Downarrow\nu"] & B^{\blacktriangle} \end{tikzcd}$$

Equivalently:

- $\exists$ an adjunction $B^{\blacktriangle} \underset{\mathrm{tot}}{\overset{\mathrm{const}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} B$ in $\mathrm{ho}\mathbf{K}$.
- $\exists$ a homotopy coherent adjunction $B^{\blacktriangle} \underset{\mathrm{tot}}{\overset{\mathrm{const}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} B$ in $\mathbf{K}$.

Totalizations of split augmented cosimplicial objects

Theorem. In any $(\infty, 2)$-category with cotensors, the totalization of a split augmented cosimplicial object is its augmentation., i.e.,

$$\begin{tikzcd} & B \arrow[d, "\mathrm{const}"] \\ B^{\mathbb{\Delta}_\infty} \arrow[ur, "\mathrm{ev}_0"] \arrow[r, "\mathrm{res}"'] \arrow[r, phantom, shift left=3, "\Downarrow\kappa"] & B^{\mathbb{\Delta}} \end{tikzcd}$$

is an absolute right lifting diagram for any object B.

Proof: is all in the weights! $\begin{tikzcd} & \mathbb{1} \arrow[dl, "{[0]}"'] \\ \mathbb{\Delta}_\infty & \mathbb{\Delta} \arrow[l, "\mathrm{incl}"] \arrow[u, "!"'] \arrow[l, phantom, shift right=3, "\Downarrow\kappa"] \end{tikzcd}$ is an equationally witnessed absolute right extension diagram in $\mathbf{Cat}$. Apply $B^{(-)}$.

Canonical presentations of algebras and descent data

Theorem. Any descent datum is the totalization of a canonical cosimplicial object of free descent data:

$$\begin{tikzcd} & \mathrm{dsc}B \arrow[d, "\mathrm{const}"] \\ \mathrm{dsc}B \arrow[ur, "\mathrm{id}"] \arrow[r, "\mathrm{can}"'] \arrow[ur, phantom, "\Downarrow\kappa"] & \mathrm{dsc}B^{\Delta} \end{tikzcd}$$

is an absolute right lifting diagram.

Theorem. Any algebra is the geometric realization of a canonical simplicial object of free algebras.:

$$\begin{tikzcd} & \mathrm{alg}B \arrow[d, "\mathrm{const}"] \\ \mathrm{alg}B \arrow[ur, "\mathrm{id}"] \arrow[r, "\mathrm{can}"'] \arrow[ur, phantom, "\Uparrow\kappa"] & \mathrm{alg}B^{\Delta^{\mathrm{op}}} \end{tikzcd}$$

is an absolute left lifting diagram.

Monadic descent in an $(\infty, 2)$-category

Theorem. For any homotopy coherent monad in an $(\infty, 2)$-category with cellular weighted limits, there is a canonical map

$$\begin{tikzcd}
B \arrow[rr, dashed] \arrow[dr] & \arrow[, phantom, "\top"] & \mathrm{dsc}B \arrow[ll, dashed, bend right] \arrow[dl] \\
& \mathrm{alg}B \arrow[ul, bend left, "\dashv"] \arrow[ur, bend right, "\vdash"'] &
\end{tikzcd}$$

- that admits a right adjoint if B has totalizations
- that is full and faithful if elements of B are totalizations of their monad resolution
- that is an equivalence if comonadicity is satisfied

The theory of comonadic codescent is dual: replace the weights by their opposites.

Monadicity theorem in an $(\infty, 2)$-category

Theorem. For any homotopy coherent adjunction $f \dashv u$ with homotopy coherent monad t, there is a canonical map

$$\begin{tikzcd}
A \arrow[rr, dashed] \arrow[dr, "u", shift left] \arrow[rr, phantom, bend left=20, "\perp"] & & \mathrm{alg}B \arrow[ll, dashed, bend right=40] \arrow[dl, "u^t", shift left] \\
& B \arrow[ul, "f", shift left] \arrow[ur, "f^t", shift left] \arrow[ul, phantom, "\perp"] \arrow[ur, phantom, "\perp"] &
\end{tikzcd}$$

- that admits a left adjoint if A has geometric realizations of u-split simplicial objects
- that is an adjoint equivalence if u creates these colimits.

Further reading

“The 2-category theory of quasi-categories” `arXiv:1306.5144`

“Homotopy coherent adjunctions and the formal theory of monads” `arXiv:1310.8279`

“A weighted limits proof of monadicity” on the n-Category Café