TOWARD THE FORMAL THEORY OF (∞, n)-CATEGORIES

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(JOINT WORK WITH DOMINIC VERITY)

Abstract. "Formal category theory" refers to a commonly applicable framework (i) for defining standard categorical structures—monads, adjunctions, limits, the Yoneda embedding, Kan extensions—and (ii) in which the classical proofs can be used to establish the expected relationships between these notions: e.g. that right adjoints preserve limits. One such framework is a 2-category equipped with a bicategory of "modules." (A *module* or *profunctor* from a category A to a category B is a functor $A^{\mathrm{op}} \times B \to \mathbf{Set}$, for instance $\hom\colon A^{\mathrm{op}} \times A \to \mathbf{Set}$.)

In [RV1], we show the basic category theory of quasi-categories can be developed formally in a strict 2-category, the "homotopy 2-category" of quasi-categories. A main point is that certain weak 2-limits present in this 2-category, particularly *comma objects*, encode universal properties up to the appropriate notion of equivalence for quasi-categories. An important feature of these "formal" definitions and proofs is that they apply representably in other higher homotopical contexts, including Rezk objects (e.g., complete Segal spaces). In the quasi-categorical context, we are reprising the foundational work pioneered by Joyal, Lurie, and others. Our aim is to develop new tools to prove further theorems, but an important side benefit is that this work applies equally to other models.

The aforementioned comma objects are precisely those modules that are represented by ordinary functors. In work in progress, we have developed a general theory of modules between quasi-categories, which is robust enough to support a complete formal category theory. (Modules appear under the guise of *correspondences* in Lurie's work, but our presentation, as *two-sided discrete fibrations*, is different.) This allows us to prove, for instance, the familiar (co)limit formula for pointwise Kan extensions.

At present, these new results do not immediately translate to other flavors of $(\infty, 1)$-categories, or to (∞, n)-categories, because there is one key technical property (a "homotopy exponentiability" criterion for maps) that we prove in specific reference to the quasi-categorical model. In what follows, we explain some of the basic ideas behind formal category theory (how comma objects are used to encode categorical structures), describe the "homotopy expontentiability" criterion, and explore future vistas.

Overview

Speaking loosely, an (∞, n)-category is a weak higher category with objects; 1-morphisms, 2-morphisms, 3-morphisms, and so on in each positive dimension; and with the property that all morphisms above dimension n are weakly invertible. For instance, the points in a space can be regarded as the objects of an $(\infty, 0)$-category whose 1-morphisms are paths, 2-morphisms are homotopies, 3-morphisms are homotopies between homotopies, and so on. Thanks to a lot of hard work

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Date: September 15, 2014 — Topologie Workshop — Oberwolfach.

with contributions made by many people, there are a plethora of models of (∞, n)-categories, which, by even more hard work, have been shown to be equivalent, in a suitable sense.

Our aim is to provide the tools necessary to do something with (∞, n)-categories: to understand what it means for some (∞, n)-category to have limits of a particular shape and what it means for a functor between (∞, n)-categories to have a right adjoint. These notions are not unrelated: as a test that the right definitions have been identified, one should be able to prove that right adjoints preserve limits.

This sort of work has been done for one particular model of $(\infty, 1)$-categories, namely the *quasi-categories* studied by Joyal, which are called ∞-*categories* by Lurie. However, the definitions and proofs are highly technical and cannot easily be generalized to other models of $(\infty, 1)$-categories or to (∞, n)-categories. To achieve this, we use ideas from formal category theory to guide our definitions and simplify our proofs. In so doing, we recover the category theory of quasi-categories and can immediately generalize most of our results to other higher category contexts.

Basic formal category theory

The simplest framework for formal category theory is a strict 2-category that we will denote by $\mathcal{K}_2$. The prototypical example might be the 2-category of categories, functors, and natural transformations. Soon $\mathcal{K}_2$ will be a 2-category whose objects are (∞, n)-categories, whose morphisms are functors of such, and whose 2-cells are homotopy classes of 1-simplices in appropriate hom-spaces, which we call natural transformations.

Adjunctions and equivalences. We will use single arrows to denote 1-cells and double arrows to denote 2-cells.

Definition. An *adjunction* consists of objects A, B; 1-cells $u\colon A \to B$, $f\colon B \to A$; and 2-cells $\eta\colon \mathrm{id}_B \Rightarrow uf$, $\epsilon\colon fu \Rightarrow \mathrm{id}_A$ satisfying the following pair of identities.

$$\begin{tikzcd}
B \arrow[rr, equal] \arrow[dr, "f"'] & {} \arrow[d, phantom, "\Downarrow\eta"] & B \arrow[dr, "f"] & \\
& A \arrow[ur, "u"'] \arrow[rr, equal] & {} \arrow[u, phantom, "\Downarrow\epsilon"] & A
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
B \arrow[d, "f"] \\
A
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
& B \arrow[rr, equal] \arrow[dr, "f"] & {} \arrow[d, phantom, "\Downarrow\eta"] & B \\
A \arrow[ur, "u"] \arrow[rr, equal] & {} \arrow[u, phantom, "\Downarrow\epsilon"] & A \arrow[ur, "u"'] &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
B \\
A \arrow[u, "u"']
\end{tikzcd}$$

In a strict 2-category, any *pasting diagram* of 2-cells, such as displayed in the preceding definition, has a unique composite 2-cell. The displayed pasting diagram is the 2-categorical encoding of the so-called triangle identities, which assert that both pasting composites of the unit and counit 2-cells are identities.

The standard proofs of the following results can be interpreted in a generic 2-category.

Proposition.
(i) *Any two left adjoints to a common 1-cell are isomorphic.*
(ii) *Adjunctions compose.*

In any 2-category, there is a standard notion of equivalence between objects.

Definition. An *equivalence* between a pair of objects A and B consists of a pair of 1-cells $u\colon A \to B$, $f\colon B \to A$ and a pair of 2-cell isomorphisms $\mathrm{id}_B \cong uf$ and $\mathrm{id}_A \cong fu$.

Again, the standard proof can be interpreted in a generic 2-category.

Proposition. *Any equivalence can be promoted to an adjoint equivalence.*

Limits and colimits. Now suppose that $\mathcal{K}_2$ has some notion of "exponentiation," an operation that defines 2-functors $(-)^X : \mathcal{K}_2 \to \mathcal{K}_2$ indexed by objects X in some other category (such as **Cat** or **sSet**). If A is an object of $\mathcal{K}_2$, the object A^X is thought of as the object of X-shaped diagrams in A. As is standard, a morphism $X \to Y$ should induce a map $A^Y \to A^X$. We'll assume also that exponentiation by the terminal object is the identity; this gives rise to a "constant diagram map" $\mathrm{const}\colon A \to A^X$ for any diagram shape X. In the examples we will consider, the provenance of such a structure is obvious, so I won't spell out precisely what is necessary.

Definition. An object $A \in \mathcal{K}_2$ *has X-shaped limits* if the 1-cell $\mathrm{const}\colon A \to A^X$ has a right adjoint $\lim\colon A^X \to A$.

Proposition. *If A and B have X-shaped limits, any right adjoint $u\colon A \to B$ preserves them.*

Proof. Suppose f is left adjoint to u. By bifunctoriality of exponentiation, the 1-cells $B \xrightarrow{f} A \xrightarrow{\mathrm{const}} A^X$ and $B \xrightarrow{\mathrm{const}} B^X \xrightarrow{f^X} A^X$ agree. Thus the right adjoints $A^X \xrightarrow{\lim} A \xrightarrow{u} B$ and $A^X \xrightarrow{u^X} B^X \xrightarrow{\lim} B$ are isomorphic. $\square$

Remark. The fully general version of the previous proposition — in which A is assumed to have certain, but not necessarily all, X-shaped limits, and nothing is assumed a priori about B — is no more difficult to prove. The reason we have not presented it here is it requires us to discuss the general definition of what it means for A to have colimits of particular X-shaped diagrams, a property which is encoded by an *absolute right lifting diagram* in $\mathcal{K}_2$. Absolute right lifting diagrams are very easy to work with but are most likely unfamiliar.

Weak comma objects. Further results are possible if the 2-category $\mathcal{K}_2$ has *weak comma objects.*

Definition. Given $B \xrightarrow{f} A \xleftarrow{g} C$, a *weak comma object* consists of the data

$$\begin{tikzcd} & f \downarrow g \arrow[dl, "\mathrm{cod}"'] \arrow[dr, "\mathrm{dom}"] & \\ C \arrow[dr, "g"'] \arrow[rr, phantom, "\Leftarrow"] & & B \arrow[dl, "f"] \\ & A & \end{tikzcd}$$

satisfying a weak universal property with three components:

(1-cell induction): Given any 2-cell as displayed on the left

$$\begin{tikzcd} & X \arrow[dl, "c"'] \arrow[dr, "b"] & \\ C \arrow[dr, "g"'] \arrow[rr, phantom, "\Leftarrow"] & & B \arrow[dl, "f"] \\ & A & \end{tikzcd} \quad = \quad \begin{tikzcd} & X \arrow[d] & \\ & f \downarrow g \arrow[dl, "\mathrm{cod}"'] \arrow[dr, "\mathrm{dom}"] & \\ C \arrow[dr, "g"'] \arrow[rr, phantom, "\Leftarrow"] & & B \arrow[dl, "f"] \\ & A & \end{tikzcd}$$

there exists a 1-cell $X \to f \downarrow g$ so that the given 2-cell factors through the comma cone along this map.

(2-cell induction): Defined similarly.

(2-cell conservativity): Any 2-cell induced by a pair of 2-cell isomorphisms is itself an isomorphism.

As a corollary of these universal properties, the 1-cells $X \to f \downarrow g$ induced by a given 2-cell form a connected groupoid over the identities on c and b.

Example. The following special cases of weak comma objects are particularly important:

$$\begin{tikzcd}
& f\downarrow A \arrow[dl, "\mathrm{cod}"'] \arrow[dr, "\mathrm{dom}"] & \\
A \arrow[rr, phantom, "\Leftarrow\alpha"] \arrow[dr, equal] & & B \arrow[dl, "f"] \\
& A &
\end{tikzcd}
\qquad
\begin{tikzcd}
& B\downarrow u \arrow[dl, "\mathrm{cod}"'] \arrow[dr, "\mathrm{dom}"] & \\
A \arrow[rr, phantom, "\Leftarrow\beta"] \arrow[dr, "u"'] & & B \arrow[dl, equal] \\
& B &
\end{tikzcd}$$

A *generalized element* of $f \downarrow A$, meaning a morphism $X \to f \downarrow A$, corresponds to a generalized element $a\colon X \to A$ of A, a generalized element $b\colon X \to B$ of B, together with a 2-cell $fb \Rightarrow a$.

Proposition. *If $f \dashv u$, then $f \downarrow A$ and $B \downarrow u$ are equivalent over $A \times B$.*

Given a morphism $fb \Rightarrow a$, there is a standard formula for the adjunct morphism of $b \Rightarrow ua$ in terms of the unit and counit of the adjunction. This argument is precisely encoded by the following 2-categorical proof.

Proof. By 1-cell induction, there are 1-cells $w\colon B \downarrow u \to f \downarrow A$ and $w'\colon f \downarrow A \to B \downarrow u$ defined to satisfy the identities:

$$\begin{tikzcd}
& B\downarrow u \arrow[dl, "\mathrm{cod}"'] \arrow[dr, "\mathrm{dom}"] \arrow[d, phantom, "\Leftarrow\beta"] & \\
A \arrow[rr, "u"] \arrow[dr, equal] & \arrow[d, phantom, "\Leftarrow\epsilon"] & B \arrow[dl, "f"] \\
& A &
\end{tikzcd}
=
\begin{tikzcd}
& B\downarrow u \arrow[d, "w"] & \\
& f\downarrow A \arrow[dl, "\mathrm{cod}"'] \arrow[dr, "\mathrm{dom}"] \arrow[d, phantom, "\Leftarrow\alpha"] & \\
A & & B \arrow[ll, "f"]
\end{tikzcd}
\qquad
\begin{tikzcd}
& f\downarrow A \arrow[dl, "\mathrm{cod}"'] \arrow[dr, "\mathrm{dom}"] \arrow[d, phantom, "\Leftarrow\alpha"] & \\
A \arrow[dr, "u"'] & \arrow[d, phantom, "\Leftarrow\eta"] & B \arrow[ll, "f"'] \arrow[dl, equal] \\
& B &
\end{tikzcd}
=
\begin{tikzcd}
& f\downarrow A \arrow[d, "w'"] & \\
& B\downarrow u \arrow[dl, "\mathrm{cod}"'] \arrow[dr, "\mathrm{dom}"] \arrow[d, phantom, "\Leftarrow\beta"] & \\
A \arrow[rr, "u"'] & & B
\end{tikzcd}$$

By definition, these 1-cells lie over $A \times B$. By the series of pasting identities

$$\begin{tikzcd}
& f\downarrow A \arrow[d, "w'"] & \\
& B\downarrow u \arrow[d, "w"] & \\
& f\downarrow A \arrow[dl, "\mathrm{cod}"'] \arrow[dr, "\mathrm{dom}"] \arrow[d, phantom, "\Leftarrow\alpha"] & \\
A & & B \arrow[ll, "f"]
\end{tikzcd}
=
\begin{tikzcd}
& f\downarrow A \arrow[d, "w'"] & \\
& B\downarrow u \arrow[dl, "\mathrm{cod}"'] \arrow[dr, "\mathrm{dom}"] \arrow[d, phantom, "\Leftarrow\beta"] & \\
A \arrow[rr, "u"] \arrow[dr, equal] & \arrow[d, phantom, "\Leftarrow\epsilon"] & B \arrow[dl, "f"] \\
& A &
\end{tikzcd}
=
\begin{tikzcd}
& f\downarrow A \arrow[dl, "\mathrm{cod}"'] \arrow[dr, "\mathrm{dom}"] \arrow[d, phantom, "\Leftarrow\alpha"] & \\
A \arrow[dr, "u"'] \arrow[dd, equal] & \arrow[d, phantom, "\Leftarrow\eta"] & B \arrow[ll, "f"'] \arrow[dl, equal] \\
\arrow[r, phantom, "\Leftarrow\epsilon"] & B \arrow[dl, "f"] & \\
A & &
\end{tikzcd}
=
\begin{tikzcd}
& f\downarrow A \arrow[dl, "\mathrm{cod}"'] \arrow[dr, "\mathrm{dom}"] \arrow[d, phantom, "\Leftarrow\alpha"] & \\
A & & B \arrow[ll, "f"]
\end{tikzcd}$$

ww' and $\mathrm{id}_{f\downarrow A}$ are induced by the same 2-cell, so there is an isomorphism between them fibered over $A \times B$. Interchanging the pair of comma objects proves similarly that $w'w \cong \mathrm{id}_{B\downarrow u}$. □

The homotopy 2-category

The strict 2-categories $\mathcal{K}_2$ of interest arise as the *homotopy 2-category* of a *quasi-categorical context* $\mathcal{K}$. A *quasi-categorical context*, is a simplicially enriched category whose hom-spaces are quasi-categories. A *quasi-category* is a simplicial set with a weak composition of morphisms in all dimensions. Part of the point of this talk is that the precise definition of this notion does not matter.

A quasi-categorical context comes equipped with two distinguished classes of maps, which we call *equivalences* and *isofibrations*. The totality of this data satisfies the properties enjoyed by the fibrant objects in a model category that is enriched over simplicial sets (with the Joyal model structure) and in which all fibrant objects are cofibrant.

The prototypical example is given by the category of quasi-categories: the Joyal model structure is monoidal. Other examples include complete Segal spaces or more general categories of *Rezk objects*: simplicial objects in a model category that are Reedy fibrant and satisfy the Segal and completeness conditions. For instance, Barwick's n-fold complete Segal space model of (∞, n)-categories has this form. If $\mathcal{K}$ is a quasi-categorical context, so is the slice category $\mathcal{K}/A$ over any object A.

Remark. Without too much additional difficulty, the hypothesis that "all of the fibrant objects are cofibrant" could be removed, but we don't know of any interesting examples in which this is not the case.

Definition. The *homotopy 2-category* $\mathcal{K}_2$ of a quasi-categorical context $\mathcal{K}$ is the strict 2-category defined by applying the homotopy category functor $\mathrm{ho}\colon \mathbf{qCat} \to \mathbf{Cat}$ to each of the hom-spaces in $\mathcal{K}$. Its objects are the objects of $\mathcal{K}$; its 1-cells are the morphisms of $\mathcal{K}$, which we call *functors*; and its 2-cells are homotopy classes of 1-simplices in the hom-spaces of $\mathcal{K}$.

A quasi-categorical context admits exponentials by arbitrary simplicial sets, and this structure descends to 2-functors on the homotopy 2-category of the form described above. Moreover, the homotopy 2-category has weak comma objects $f \downarrow g$ defined by the pullback

$$\begin{tikzcd} f \downarrow g \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & A^{\Delta^1} \arrow[d, two heads] \\ C \times B \arrow[r, "g\times f"'] & A \times A \end{tikzcd}$$

in $\mathcal{K}$. This pullback exists because the fibrant objects in a simplicial model category are closed under Bousfield-Kan-style homotopy limits. More precisely, the comma construction is an example of a simplicially enriched (weighted) limit of a pointwise fibrant diagram whose weight is projective cofibrant.

Formal category theory in a quasi-categorical context. The content of the papers [RV1, RV2, RV3] is stated in the language of quasi-categories but all of the results appearing there apply, essentially without change, in any quasi-categorical context $\mathcal{K}$. What this means is that the definitions of the basic categorical concepts can be interpreted in $\mathcal{K}$ and the proofs, largely taking place in the homotopy 2-category, are also unchanged.

For example, in [RV2], we define the object of algebras for a homotopy coherent monad to be a particular projective cofibrant weighted limit and prove the monadicity theorem, which characterizes the accompanying free-forgetful adjunction in $\mathcal{K}_2$.

Two-sided discrete fibrations

Perhaps the most important thing that is missing from the basic framework of a 2-category with weak comma objects is the Yoneda embedding (classically, the "hom" bifunctor $A^{\mathrm{op}} \times A \to \mathbf{Set}$) and its generalizations (arbitrary functors $B^{\mathrm{op}} \times A \to \mathbf{Set}$). These go by a variety of names: *modules*, *profunctors*, *distributors*, or *correspondences*. There are several possible ways to encode modules in a 2-category. Given the structures that are present in a homotopy 2-category $\mathcal{K}_2$, our preference will be to use weak comma objects.

For example, the Yoneda embedding for A is encoded by the comma object:

$$\begin{tikzcd} & A \downarrow A \arrow[dl, "\mathrm{cod}"'] \arrow[dr, "\mathrm{dom}"] & \\ A \arrow[dr, equal] \arrow[rr, phantom, "\Leftarrow"] & & A \arrow[dl, equal] \\ & A & \end{tikzcd}$$

By 1-cell induction, any 2-cell with codomain A can be encoded by a functor abutting to $A \downarrow A$.

$$\begin{tikzcd} X \arrow[d, bend right=50] \arrow[d, bend left=50] \arrow[d, phantom, "\Leftarrow"] \\ A \end{tikzcd} \quad = \quad \begin{tikzcd} X \arrow[d] \\ A \downarrow A \arrow[d, bend right=50, "\mathrm{cod}"'] \arrow[d, bend left=50, "\mathrm{dom}"] \arrow[d, phantom, "\Leftarrow"] \\ A \end{tikzcd}$$

But $A \downarrow A$ has additional universal properties relating to the pre- and post-composition actions by arrows in A, which can only partially be established by 2-cell induction. These additional universal properties are expressed by saying that $A \xleftarrow{\mathrm{cod}} A \downarrow A \xrightarrow{\mathrm{dom}} A$ is a *two-sided discrete fibration* in $\mathcal{K}_2$.

Cartesian fibrations. To state this definition, we first need a notion of cartesian fibration. For quasi-categories, this coincides exactly with the notion introduced by Lurie, but our 2-categorical definition can be interpreted in any homotopy 2-category.

Definition. An isofibration $p\colon E \twoheadrightarrow B$ is a *cartesian fibration* if

(i) Every $\begin{tikzcd} X \arrow[r, "e"] \arrow[dr, "b"'] & E \arrow[d, "p"] \arrow[dl, phantom, "\Uparrow\alpha"] \\ & B \end{tikzcd}$ admits a *p-cartesian lift* $\chi\colon \bar{e} \Rightarrow e$ along p. Here a 2-cell χ is p-cartesian if it satisfies a weak form of the expected factorization axiom and also has a 2-cell conservativity property: any endomorphism of χ siting over the identity on b is an isomorphism.

(ii) The p-cartesian 2-cells are stable under restriction along any functor.

Replacing the 2-category $\mathcal{K}_2$ by its dual $\mathcal{K}_2^{\mathrm{co}}$, which reverses the 2-cells but not the 1-cells, we obtain the notion of a *cocartesian fibration.* The search for examples is greatly aided by the following theorem, which is a generalization of a 2-categorical result of Street [S].

Theorem. *For an isofibration $p\colon E \twoheadrightarrow B$, the following are equivalent:*

(i) *p is a cartesian fibration.*

(ii) *The natural functor*

$$\begin{tikzcd} E \arrow[rr] \arrow[dr, twoheadrightarrow, "p"'] & & B \downarrow p \arrow[dl, twoheadrightarrow, "dom"] \\ & B & \end{tikzcd}$$

admits a right adjoint over B.

(iii) *The natural functor $E \downarrow E \to B \downarrow p$ admits a right adjoint right inverse.*

The right adjoint in (ii) picks out the domain of the lifted 2-cells; the p-cartesian lift is specified by the counit. The functor in (iii) picks out the p-cartesian lift.

Example. Using this theorem and the universal properties of weak comma objects, it is relatively straightforward to show that dom: $E \downarrow E \twoheadrightarrow E$ is a cartesian fibration. More generally, dom: $f \downarrow g \to B$ is a cartesian fibration.

Definition. A cartesian fibration $p\colon E \twoheadrightarrow B$ is *discrete* if any 2-cell $\begin{tikzcd} X \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Uparrow"] & E \end{tikzcd}$ whose composite with p is an identity is an isomorphism.

Example. Let $1 \in \mathcal{K}_2$ denote the terminal object. If $b\colon 1 \to B$ is a point, then dom: $B \downarrow b \twoheadrightarrow B$ is a discrete cartesian fibration.

Finally:

Definition. A span $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$ is a *two-sided discrete fibration* if

(i) $E \to A \times B$ is a discrete cartesian fibration in $\mathcal{K}_2/A$.

(ii) $E \to A \times B$ is a discrete cocartesian fibration in $\mathcal{K}_2/B$.

Example. The span $C \xleftarrow{\text{cod}} f \downarrow g \xrightarrow{\text{dom}} B$ is a two-sided discrete fibration.

The equipment for quasi-categories

With the notion of a two-sided discrete fibration to encode modules, we can now introduce the complete framework for formal category theory.

Theorem. *There is a bicategory* $\mathbf{qMod}_2$ *of quasi-categories; modules, i.e., two-sided discrete fibrations $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$, written $E\colon A \nrightarrow B$; and isomorphism classes of maps of spans. Moreover,* $\mathbf{qMod}_2$ *is biclosed: the functors $E \otimes_B -$ and $- \otimes_A E$ admit right biadjoints.*

Note that the proposition proven above asserts that if $f \dashv u$, then the modules $f \downarrow A$ and $B \downarrow u$ are isomorphic as 1-cells $A \nrightarrow B$.

Remark. For this result, and from hereon, we have specialized to the case of quasi-categories. Twice in its proof, we use the fact that (co)cartesian fibrations are "homotopy exponentiable": that pullback along a (co)cartesian fibration $p\colon E \twoheadrightarrow B$ defines a left Quillen functor $p^*\colon \mathbf{sSet}/B \to \mathbf{sSet}/E$. As in category theory, not all functors have this property, but there is a particular *conduché condition*, asserting that a certain family of simplicial sets are weakly contractible, that implies it. We have yet to explore how it may be generalized to other contexts. For the time being, we might note that the structures on $\mathbf{qMod}_2$ requiring this condition are convenient, but not strictly necessary.

The bicategory $\mathbf{qMod}_2$ extends the homotopy 2-category of quasi-categories:

Theorem. *The identity-on-objects homomorphism* $\mathbf{qCat}_2 \hookrightarrow \mathbf{qMod}_2$ *that carries* $f\colon A \to B$ *to* $B \downarrow f\colon A \nrightarrow B$ *is locally fully faithful.*

The proof of this result uses the Yoneda lemma for maps between modules.

Theorem. *The covariant represented module* $B \downarrow f\colon A \nrightarrow B$ *is left adjoint to the contravariant represented module* $f \downarrow B\colon B \nrightarrow A$ *in* $\mathbf{qMod}_2$*.*

The preceding three theorems combine to assert that $\mathbf{qCat}_2 \hookrightarrow \mathbf{qMod}_2$ defines an *equipment* for quasi-categories, in the sense of Wood [W].

Formal category theory. With this structure in place, we can now commence with the formal category theory. For example, there is a standard definition of a right (Kan) extension diagram in any 2-category. In $\mathbf{qCat}_2$ this is too weak (failing, in general, to be "pointwise"), but in $\mathbf{qMod}_2$ it gives the correct notion.

Definition. Consider a pair of functors $f\colon A \to B$ and $g\colon A \to C$. Because $\mathbf{qMod}_2$ is closed, we can form the following right extension diagram

$$\begin{tikzcd} A \arrow[rr, "C\downarrow g", "|" marking] \arrow[dr, "B\downarrow f"', "|" marking] & \arrow[d, phantom, "\Leftarrow"] & C \arrow[dl, dashed, "E", "|" marking] \\ & B & \end{tikzcd}$$

If the module E is covariantly representable, i.e., if $E \cong B \downarrow r$ for some $r\colon C \to B$, then r is the *right extension of* f *along* g.

We use the following theorem to identify representable modules, which allows us to determine when right extensions exist.

Theorem. *A module* $C \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$ *is covariant representable if and only if the following equivalent conditions hold:*

(i) *q has a right adjoint right inverse*
(ii) *each fiber of q has a terminal object*

The right inverse of (i), composed with p, defines the representing functor $C \to B$. Condition (ii) can be used to establish the expected result: if B has limits indexed by certain comma objects, then the right extension of f along g exists.

Epilogue: the conduché condition

Consider an isofibration $p\colon E \twoheadrightarrow B$ between quasi-categories. We want a condition that implies that $p^*\colon \mathbf{sSet}/B \to \mathbf{sSet}/E$ is left Quillen with respect to the Joyal model structures.

Definition. p is *conduché* if for all

$$\begin{tikzcd} \Delta^{n-1} \arrow[r, "d^k"] \arrow[d, "e"'] & \Delta^n \arrow[d, "b"] \\ E \arrow[r, two heads, "p"'] & B \end{tikzcd} \qquad 0 < k < n,$$

the conduché space $C_p(e, b, k)$ is weakly contractible.

The conduché space $C_p(e, b, k)$ is a simplicial set whose vertices are those n-simplices of E lying over b and with kth face e. The higher simplices are defined similarly but with respect to degenerated copies of b at the kth vertex.

For any isofibration p, the proof that p^* is left Quillen immediately reduces to the question of whether the monomorphism $Y \to X$ defined by pullback

$$\begin{tikzcd}
Y \arrow[rr] \arrow[dr, tail] \arrow[ddr] & & \Lambda^n_k \arrow[dr] \arrow[ddr] & \\
& X \arrow[rr] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & & \Delta^n \arrow[d, "b"] \\
& E \arrow[rr, two heads, "p"'] & & B
\end{tikzcd}$$

is a trivial cofibration in the Joyal model structure. Note that the vertices e indexing the conduché spaces correspond to $(n-1)$-simplices that are present in X but missing from Y. The 0-simplices of $C_p(e, b, k)$ are the n-simplices that could be attached to a Λ^n_k-horn in Y to adjoin e. The proof that pulling back along a conduché functor is left Quillen uses a Reedy category argument to present $Y \to X$ as a relative cell complex built from inner horn inclusions indexed by data derived from the conduché spaces.

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