# Made-to-Order Weak Factorization Systems

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## 1 The Algebraic Small Object Argument

For a cocomplete category **M** which satisfies certain "smallness" condition (such as being locally presentable), the *algebraic small object argument* defines the functorial factorization necessary for a "made-to-order" weak factorization system with right class $\mathcal{J}^{\boxslash}$. For now, $\mathcal{J}$ is an arbitrary set of morphisms of **M** but later we will use this notation to represent something more sophisticated.

The small object argument begins by defining a *generic lifting problem*, a single lifting problem that characterizes the desired right class:

$$f \in \mathcal{J}^{\boxslash} \quad \leftrightsquigarrow \quad \begin{tikzcd} \cdot \arrow[r] \arrow[d, "\coprod_{j \in \mathcal{J}} \coprod_{\mathrm{Sq}(j,f)} j"'] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r] \arrow[ur, dashed] & \cdot \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \cdot \arrow[r] \arrow[d, "\coprod_{j \in \mathcal{J}} \coprod_{\mathrm{Sq}(j,f)} j"'] & \cdot \arrow[r, equal] \arrow[d, "L_1 f"] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \cdot \arrow[r, "R_1 f"'] \arrow[ur, dashed, "s"'] & \cdot \end{tikzcd} \tag{1}$$

The diagonal map defines a solution to any lifting problem between $\mathcal{J}$ and $f$. Taking a pushout transforms the generic lifting problem into the *step-one functorial factorization*, another generic lifting problem that also factors $f$.

This defines a pointed endofunctor $R_1 \colon \mathbf{M}^{\mathbf{2}} \to \mathbf{M}^{\mathbf{2}}$ of the arrow category. An $R_1$-*algebra* is a pair $(f, s)$ as displayed. By construction, $L_1 f \in {}^{\boxslash}(\mathcal{J}^{\boxslash})$. However, there is no reason to expect that $R_1 f \in \mathcal{J}^{\boxslash}$: maps in the image of $R_1$ need not be $R_1$-algebras – unless $R_1$ is a monad. The idea of the algebraic small object argument, due to Garner [5], is to freely replace the pointed endofunctor $R_1$ by a monad. (When all maps in the left class are monomorphisms, the free monad is defined by "iteratively attaching non-redundant cells" until this process converges.)

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Following Kelly [6], and assuming certain "smallness" or "boundedness" conditions, it is possible to construct the free monad $\mathbb{R}$ on a pointed endofunctor $R_1$ in such a way that the categories of algebras are isomorphic. Garner shows that with sufficient care, Kelly's construction can be performed in a way that preserves the fact that the endofunctor $R_1$ is the right factor of a functorial factorization whose left factor $L_1$ is already a comonad. In this way, the algebraic small object produces a functorial factorization $f = \mathbb{R}f \cdot \mathbb{L}f$ in which $\mathbb{L}$ is a comonad, $\mathbb{R}$ is a monad, and $\mathbb{R}$-**Alg** $\cong R_1$-**Alg** $\cong \mathcal{J}^{\boxslash}$.

*Example 1* Consider $\{\emptyset \to *\}$ on the category of sets. The algebraic small object argument produces the generic lifting problem displayed on the left and the step-one functorial factorization displayed on the right:

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y \arrow[r, equal] \arrow[ur, dashed] & Y \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \emptyset \arrow[r] \arrow[d] & X \arrow[r, equal] \arrow[d, "\mathrm{incl}_1"] & X \arrow[d, "f"] \\ Y \arrow[r] & X \coprod Y \arrow[r, "f \coprod 1"'] \arrow[ur, dashed] \arrow[ul, phantom, "\ulcorner", very near start] & Y \end{tikzcd}$$

Every lifting problem after step one is redundant. Indeed, $\mathbb{R}f = f \coprod 1$ is already a monad and the construction converges in one step to define the factorization $f = f \coprod 1 \cdot \mathrm{incl}_1$.

*Example 2* Consider $\{\partial\Delta^n \hookrightarrow \Delta^n\}_{n\geq 0}$ on the category of simplicial sets. Here we may consider lifting problems against a single generator at a time, inductively by dimension. The step-one factorization of $X \to Y$ attaches the 0-skeleton of $Y$ to $X$. There are no non-redundant lifting problems involving the generator $\emptyset \hookrightarrow \Delta^0$, so we move up a dimension. The step-two factorization of $X \to Y$ now attaches 1-simplices of $Y$ to all possible boundaries in $X \cup \mathrm{sk}_0 Y$. After doing so, there are no non-redundant lifting problems involving $\partial\Delta^1 \hookrightarrow \Delta^1$. The construction converges at step $\omega$.

The algebraic small object argument produces an *algebraic weak factorization system* $(\mathbb{L}, \mathbb{R})$, a functorial factorization that underlies a comonad $\mathbb{L}$ and a monad $\mathbb{R}$, and in which the canonical map $LR \Rightarrow RL$ defines a distributive law. The functorial factorization $f = \mathbb{R}f \cdot \mathbb{L}f$ characterizes the underlying weak factorization system $(\mathcal{L}, \mathcal{R})$:

$$f \in \mathcal{L} \quad \leftrightsquigarrow \quad \begin{tikzcd} \cdot \arrow[r, "Lf"] \arrow[d, "f"'] & \cdot \arrow[d, "Rf"] \\ \cdot \arrow[r, equal] \arrow[ur, dashed, "s"'] & \cdot \end{tikzcd} \qquad\qquad g \in \mathcal{R} \quad \leftrightsquigarrow \quad \begin{tikzcd} \cdot \arrow[r, equal] \arrow[d, "Lg"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "Rg"'] \arrow[ur, dashed, "t"] & \cdot \end{tikzcd}$$

because the specified lifts assemble into a canonical solution to any lifting problem:

$$
\begin{tikzcd}
\cdot \arrow[r, "u"] \arrow[d, "f"'] & \cdot \arrow[d, "g"] \\
\cdot \arrow[r, "v"'] & \cdot
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
\cdot \arrow[r, "u"] \arrow[d, "Lf"'] & \cdot \arrow[d, "Lg"] \\
\cdot \arrow[r, dashed] \arrow[d, "Rf"'] \arrow[ur, dashed, "t"] & \cdot \arrow[d, "Rg"] \\
\cdot \arrow[r, "v"'] \arrow[u, dashed, bend left, "s"] & \cdot
\end{tikzcd}
$$

## 2 Generalizations of the Algebraic Small Object Argument

The construction of the generic lifting problem admits a more categorical description which makes it evident that it can be generalized in a number of ways, expanding the class of weak factorization systems whose functorial factorizations can be "made-to-order."

Step zero of the algebraic small object argument forms the *density comonad*, i.e., the left Kan extension along itself, of the inclusion of the generating set of arrows:

$$
\begin{tikzcd}
 & \mathbf{M}^2 \arrow[dr, dashed, "\mathrm{Lan}=L_0"] \arrow[d, phantom, "\Uparrow"] & \\
\mathcal{J} \arrow[ur] \arrow[rr] & {} & \mathbf{M}^2
\end{tikzcd}
\qquad
L_0 f = \int^{j \in \mathcal{J}} \coprod_{\mathrm{Sq}(j,f)} j.
$$

When **M** is cocomplete, this construction makes sense for any small *category* of arrows $\mathcal{J}$. The counit of the density comonad defines the generic lifting problem (1), admitting a solution if and only if $f \in \mathcal{J}^{\boxslash}$ – but now $\mathcal{J}^{\boxslash}$ denotes the category in which an object is a map $f$ together with a choice of solution to any lifting problem against $\mathcal{J}$ that is coherent with respect to (i.e., commutes with) morphisms in $\mathcal{J}$. Proceeding as before, the algebraic small object argument produces an algebraic weak factorization system $(\mathbb{L}, \mathbb{R})$ so that $\mathbb{R}$-**Alg** $\cong \mathcal{J}^{\boxslash}$ over $\mathbf{M}^2$, and $\mathbb{L}$-coalgebras lift against $\mathbb{R}$-algebras.

*Example 3* In the category of cubical sets, let $\sqcap, \sqsupset, \sqcup, \sqsubset$ suggestively denote four subfunctors of the 2-dimensional representable $\square^2$. For $n > 2$ and $J \subset \{1, \ldots, n\}$ with $|J| = n - 2$, define $\sqcap^J \subset \square^n$ to be $\sqcap \otimes \square^J$, and similarly for the other three shapes. Consider the category whose objects are the inclusions $\sqcap^J \hookrightarrow \square^n$ for each shape and whose morphisms are generated by

- the projections
$$\begin{tikzcd} \sqcap^J \arrow[r] \arrow[d] & \sqcap^{J\setminus\{j\}} \arrow[d] \\ \square^n \arrow[r, "\pi_j"'] & \square^{n-1} \end{tikzcd}$$
for each $j \in J$, and
- the inclusions
$$\begin{tikzcd} \sqcap^J \arrow[r] \arrow[d] & \sqcap^{J\cup\{i\}} \arrow[d] \\ \square^n \arrow[r] & \square^{n+1} \end{tikzcd}$$
embedding $\square^n$ as the face $i = 0$ or $i = 1$.

This generates the fibrant replacement functor, see Bezem–Coquand–Huber [3].

*Example 4 ([8, §4.2])* Any algebraic weak factorization system $(\mathbb{L}, \mathbb{R})$ on $\mathbf{M}$ induces a pointwise-defined algebraic weak factorization system $(\mathbb{L}^{\mathbf{A}}, \mathbb{R}^{\mathbf{A}})$ on the category $\mathbf{M}^{\mathbf{A}}$ of diagrams. Moreover, when $(\mathbb{L}, \mathbb{R})$ is generated by $\mathcal{J}$, $(\mathbb{L}^{\mathbf{A}}, \mathbb{R}^{\mathbf{A}})$ is generated by the category $\mathbf{A}^{\mathrm{op}} \times \mathcal{J}$, whose objects are tensors of arrows of $\mathcal{J}$ with covariant representables.

If $\mathbf{M}$ is tensored, cotensored, and enriched over a closed monoidal category $\mathbf{V}$, we may choose to define the generic lifting problem using the $\mathbf{V}$-enriched left Kan extension

$$L_0 f = \int^{j \in \mathcal{J}} \underline{\mathrm{Sq}}(j,f) \otimes j,$$

where $\underline{\mathrm{Sq}}(j,f) \in \mathbf{V}$ is the object of commutative squares. The enriched algebraic small object argument produces an algebraic weak factorization system whose underlying left and right classes satisfy an enriched lifting property, defined internally to $\mathbf{V}$. The classes of an ordinary weak factorization system satisfy this enriched lifting property if and only if tensoring with objects from $\mathbf{V}$ preserves the morphisms in the left class [9, §13].

*Example 5* Consider $\{0 \to R\}$ in the category of modules over a commutative ring $R$ with identity. In analogy with Example 1, the unenriched algebraic small object argument produces the left-hand functorial factorization, while the enriched algebraic small object argument produces the factorization on the right:

$$X \xrightarrow{\mathrm{incl}} X \oplus (\oplus_Y R) \xrightarrow{f \oplus \mathrm{ev}} Y, \qquad X \xrightarrow{\mathrm{incl}} X \oplus Y \xrightarrow{f \oplus 1} Y.$$

*Example 6 (Barthel–May–Riehl [1])* On the category of unbounded chain complexes of $R$-modules, consider the sets $\{0 \to D^n\}_{n \in \mathbb{Z}}$ and $\{S^{n-1} \hookrightarrow D^n\}_{n \in \mathbb{Z}}$, where $D^n$ is the chain complex with $R$ in degrees $n$ and $n-1$ and identity differential, and where $S^n$ has $R$ in degree $n$ and zeroes elsewhere. The enriched algebraic small object argument converges at step one in the former case and at step two in the latter case to produce the natural factorizations through the mapping cocylinder and the mapping cylinder, respectively (see "Mapping (co)cylinder factorizations via the small object argument" on the $n$-Category Café).

The algebraic weak factorization systems constructed in Examples 4 and 6 are not cofibrantly generated (in the usual sense) [4, 7].

*Example 7 (Barthel–Riehl [2])* There are two algebraic weak factorization systems on topological spaces whose right class is the class of Hurewicz fibrations. A map is a *Hurewicz fibration* if it has the homotopy lifting property, i.e., solutions to lifting problems

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\mathrm{incl}_0"'] & X \arrow[d, "f"] \\ A \times I \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \tag{2}$$

defined for every topological space $A$. As there is proper class of generators, it is not possible to form the coproduct in (1). However, the functor $\mathbf{Top}^{\mathrm{op}} \to \mathbf{Set}$ sending $A$ to the set of lifting problems (2) is represented by the mapping cocylinder $Nf$:

$$\begin{tikzcd} Nf \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y^I \arrow[d, "\mathrm{ev}_0"] \\ X \arrow[r, "f"'] & Y \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} Nf \arrow[r] \arrow[d, "\mathrm{incl}_0"'] & X \arrow[d, "f"] \\ Nf \times I \arrow[r] & Y \end{tikzcd}$$

It follows that any lifting problem (2) factors uniquely through the generic lifting problem displayed on the right. The algebraic small object argument proceeds as usual, though there are some subtleties in the proof of its convergence.

There is another algebraic weak factorization system "found in the wild": the factorization through the space of Moore paths. The category of algebras for the Moore paths monad admits the structure of a double category in such a way that the forgetful functor to the arrow category becomes a double functor. A recognition criterion due to Garner implies that this defines an algebraic weak factorization system.

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## References


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