Functoriality in algebra and topology

Emily Riehl

Johns Hopkins University
http://www.math.jhu.edu/~eriehl

Macquarie University
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Plan

Part I. Categories of “nouns” and “verbs” and the functors between them.

Part II. Functoriality in algebraic topology: a proof of the Brouwer Fixed Point Theorem

Part III. Functoriality in data analysis: clustering algorithms

Nouns and verbs

A functor is a "meta function" between mathematical theories packaged as categories.

> *It frames a possible template for any mathematical theory: the theory should have nouns and verbs, i.e., objects, and morphisms, and there should be an explicit notion of composition related to the morphisms; the theory should, in brief, be packaged by a category.*

— Barry Mazur, "When is one thing equal to some other thing?"

Categories

> *. . . the theory should have nouns and verbs, i.e., objects, and morphisms, and there should be an explicit notion of composition related to the morphisms; the theory should, in brief, be packaged by a category.*

A category has objects (the nouns) and morphisms (the verbs).

Example.

- $\mathsf{Top}_* \begin{cases} \text{objects} & = \text{based spaces} \\ \text{morphisms} & = \text{based continuous maps} \end{cases}$
- $\mathsf{Group} \begin{cases} \text{objects} & = \text{groups} \\ \text{morphisms} & = \text{homomorphisms} \end{cases}$

Functors

A functor is a morphism between categories.

Example. $\mathsf{Top}_* \begin{cases} \text{based spaces} \\ \text{based maps} \end{cases}$ $\mathsf{Group} \begin{cases} \text{groups} \\ \text{homomorphisms} \end{cases}$
are related by the **fundamental group functor**:

$$\mathsf{Top}_* \xrightarrow{\pi_1} \mathsf{Group}$$

$$\begin{array}{ccc} (X,x) & \mapsto & \pi_1(X,x) \\ f\big\downarrow & \mapsto & \big\downarrow \pi_1 f \\ (Y,y) & \mapsto & \pi_1(Y,y) \end{array}$$

Functoriality

A mapping

$$\mathsf{Top}_* \xrightarrow{\pi_1} \mathsf{Group}$$

$$\begin{array}{ccc} (X,x) & \mapsto & \pi_1(X,x) \\ f\big\downarrow & \mapsto & \big\downarrow \pi_1 f \\ (Y,y) & \mapsto & \pi_1(Y,y) \end{array}$$

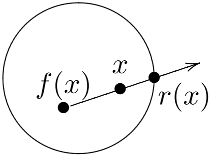
is functorial if

- $\pi_1(g \circ f) = \pi_1 g \circ \pi_1 f$ "functors preserve composition"
- $\pi_1(\mathrm{id}_{(X,x)}) = \mathrm{id}_{\pi_1(X,x)}$ "functors preserve identities"

Brouwer Fixed Point Theorem

Theorem. Any continuous endomorphism of a 2-dimensional disk D^2 has a fixed point.

Proof: Assuming $f\colon D^2 \to D^2$ is such that $f(x) \neq x$ for all $x \in D^2$, there is a continuous function $r\colon D^2 \to S^1$ that carries a point $x \in D^2$ to the intersection of the ray from $f(x)$ to x with the boundary circle S^1.



Note that the function r fixes the points on the boundary circle $S^1 \subset D^2$. Thus, r defines a retraction of the inclusion $i\colon S^1 \hookrightarrow D^2$, which is to say, the composite $S^1 \xrightarrow{i} D^2 \xrightarrow{r} S^1$ is the identity.

Brouwer Fixed Point Theorem

Theorem. Any continuous endomorphism of a 2-dimensional disk D^2 has a fixed point.

We have defined continuous functions whose composite

$$S^1 \xrightarrow{i} D^2 \xrightarrow{r} S^1$$

is the identity. Now apply the functor π_1 to obtain a composable pair of group homomorphisms:

$$\begin{tikzcd} \pi_1(S^1) \arrow[r, "\pi_1 i"] \arrow[d, equal] & \pi_1(D^2) \arrow[r, "\pi_1 r"] \arrow[d, equal] & \pi_1(S^1) \arrow[d, equal] \\ \mathbb{Z} \arrow[r] & 0 \arrow[r] & \mathbb{Z} \end{tikzcd}$$

By the functoriality axioms, we must have

$$\pi_1 r \circ \pi_1 i = \pi_1(r \circ i) = \pi_1(\mathrm{id}_{S^1}) = \mathrm{id}_{\pi_1(S^1)}\,.$$

But $0 \neq 1$, a contradiction $\square$

A decategorified interlude

A clustering algorithm is a function $\mathfrak{C}$ that converts a metric space (X, d_X) into a partition P_X of its points into "clusters."

Desired properties. Theorem (Kleinberg). There exists no clustering algorithm satisfying:

- *scale invariance*: $\mathfrak{C}(X, d_X) = \mathfrak{C}(X, \lambda d_X)$ for any $\lambda > 0$
- *surjectivity*: for any partition P_X of X there is some metric d_X on X so that $\mathfrak{C}(X, d_X) = P_X$
- *consistency*: if the metric is altered to make the clusters "more obvious" the clustering should not change

Functorial clustering schemes

> *Functoriality refers to the idea that one should be able to compare the results of clustering algorithms as one varies the dataset, for example by adding points or by applying functions to it.*

— Gunnar Carlsson and Facundo Mémoli, “Classifying clustering schemes”

Categories of metric spaces and clusters

- Metric $\begin{cases} \text{objects} & = \text{finite metric spaces} \\ \text{morphisms} & = \text{distance non-increasing maps} \end{cases}$
- Cluster $\begin{cases} \text{objects} & = \text{partitioned sets} \\ \text{morphisms} & = \text{``coalescing'' functions} \end{cases}$

A morphism $f\colon (X, P_X) \to (Y, P_Y)$ of partitioned sets is:

- a function $f\colon X \to Y$
- so that P_X refines $f^{-1}(P_Y)$

"clusters may coalesce but do not break up"

A clustering functor

A clustering functor

$$\begin{tikzcd} \mathsf{Metric} \arrow[rr, "\mathfrak{C}"] \arrow[dr, "\text{points}"'] & & \mathsf{Cluster} \arrow[dl, "\text{points}"] \\ & \mathsf{Set} & \end{tikzcd}$$

- assigns to each metric space (X, d_X) a partitioned set (X, P_X)
- so that for any distance non-increasing continuous function $f\colon (X, d_X) \to (Y, d_Y)$, the clustering scheme P_X refines $f^{-1}(P_Y)$.

Theorem (Carlsson-Mémoli). The only scale-invariant functors $\mathfrak{C}$: Metric $\to$ Cluster either assign each space the discrete partition or assign each space the singleton partition.

The problem with scale invariance

Let $\Delta^1(d)$ denote the metric space with two points of distance d.

Scale affects clustering:

- If d is massive, then even the most nearsighted clustering algorithm should be able to distinguish two distinct clusters.
- But if d is very small, then even the most eagle-eyed clustering algorithm should see a single cluster.

Theorem (Carlsson-Mémoli). If $\mathfrak{C}$: Metric $\to$ Cluster is a clustering functor for which there exists $r > 0$ so that

$$\mathfrak{C}(\Delta^1(d)) = \begin{cases} \text{two points} & d > r \\ \text{one point} & d \leq r \end{cases}$$

then $\mathfrak{C}$ is the Vietoris-Rips functor $\mathfrak{R}_r$.

The Vietoris-Rips functor

The Vietoris-Rips functor is defined by

$$\text{Metric} \xrightarrow{\mathfrak{R}_r} \text{Cluster}$$

$$(X, d_X) \mapsto (X, X_{/\sim_r})$$

where $x \sim_r x'$ if there is a finite sequence of points

$$x = x_0, x_1, \ldots, x_n = x' \qquad \text{with} \qquad d_X(x_i, x_{i+1}) \leq r.$$

Hierarchical clustering schemes

The Vietoris-Rips functor is more naturally expressed as a functor

$$\mathsf{Metric} \xrightarrow{\mathfrak{R}} \mathsf{PCluster}$$

where PCluster is the category of persistent clusters: sets X equipped with an order-preserving functor

$$\begin{array}{ccc} \mathbb{R}_+ & \longrightarrow & \mathsf{Cluster}_X \\ r & \mapsto & P_X(r) \end{array}$$

so that if $r < s$ then $P_X(r)$ refines $P_X(s)$.

"Clusters coalesce as $r > 0$ increases."

Existence and uniqueness of hierarchical clustering schemes

Theorem (Carlsson-Mémoli). If

$$\begin{tikzcd} \mathsf{Metric} \arrow[rr, "\mathfrak{C}"] \arrow[dr, "\mathrm{points}"'] & & \mathsf{PCluster} \arrow[dl, "\mathrm{points}"] \\ & \mathsf{Set} & \end{tikzcd}$$

is a hierarchical clustering functor so that

- $\mathfrak{C}(\Delta^1(d), r) = \begin{cases} \text{two points} & d > r \\ \text{one point} & d \leq r \end{cases}$
- when the parameter r is smaller than the separation of X, $\mathfrak{C}(X, r)$ is the discrete partition of X

then $\mathfrak{C} = \mathfrak{R}$ is the Vietoris-Rips functor.

Note that $\mathfrak{R}$ satisfies scale invariance, surjectivity, and is *functorial* — a property similar to Kleinberg's "consistency."

Further reading

"When is one thing equal to some other thing?" Barry Mazur, 2007

"Thinking about Grothendieck," Barry Mazur, 2016

"An impossibility theorem for clustering," Jon Kleinberg, 2002

"Classifying clustering schemes," Gunnar Carlsson and Facundo Mémoli, 2013

"Categorification," Chapter 10 of *Elementary Applied Topology*, Robert Ghrist, 2014