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The complicial sets model of higher ∞-categories

Between Topology and Quantum Field Theory
A conference in celebration of Dan Freed

The idea of a higher ∞-category

An ∞-category, a nickname for an $(\infty, 1)$-category, has:

- objects
- 1-arrows between these objects
- with composites of these 1-arrows witnessed by invertible 2-arrows
- with composition associative up to invertible 3-arrows (and unital)
- with these witnesses coherent up to invertible arrows all the way up

A higher ∞-category, meaning an (∞, n)-category for $0 \leq n \leq \infty$, has:

- objects
- 1-arrows between these objects
- 2-arrows between these 1-arrows
- $\vdots$
- n-arrows between these $n-1$-arrows
- plus higher invertible arrows witnessing composition, units, associativity, and coherence all the way up

Fully extended topological quantum field theories

The (∞, n)-category Bord_n has

- objects = compact 0-manifolds
- k-arrows = k-manifolds with corners, for $1 \leq k \leq n$
- $n + 1$-arrows = diffeomorphisms of n-manifolds rel boundary
- $n + m + 1$-arrows = m-fold isotopies of diffeomorphisms, $m \geq 1$

often with extra structure (eg framing, orientation, G-structure).

A fully extended topological quantum field theory is a homomorphism with domain Bord_n, preserving the monoidal structure and all compositions. The cobordism hypothesis classifies fully extended TQFTs of framed bordisms by the value taken by the positively oriented point.

Dan Freed

- The cobordism hypothesis, Bulletin of the AMS, vol 50, no 1, 2013, 57–92; arXiv:1210.5100

On the unicity of the theory of higher ∞-categories

The schematic idea of an (∞, n)-category is made rigorous by various models: θ_n-spaces, iterated complete Segal spaces, Segal n-categories, n-quasi-categories, n-relative categories, …

Theorem (Barwick–Schommer-Pries, et al). All of the above models of (∞, n)-categories are equivalent.

Clark Barwick and Christopher Schommer-Pries

- On the Unicity of the Homotopy Theory of Higher Categories
  arXiv:1112.0040

Thus, it's tempting to work "model independently" when envoking higher ∞-categories.

But the theory of higher ∞-categories has not yet been comprehensively developed in any model, so there is "analytic" work still to be done.

Plan

Goal: introduce a user-friendly model of higher ∞-categories

1. A simplicial model of $(\infty, 1)$-categories

2. Towards a simplicial model of $(\infty, 2)$-categories

3. The complicial sets model of higher ∞-categories

4. Complicial sets in the wild (joint with Dominic Verity)

1

A simplicial model of $(\infty, 1)$-categories

The idea of a 1-category

A 1-category has:

- objects: $\bullet$
- 1-arrows: $\bullet \longrightarrow \bullet$
- composition: $$\begin{tikzcd} \bullet \arrow[r, "f"] \arrow[rr, bend right, dashed, "g\circ f"'] & \bullet \arrow[r, "g"] & \bullet \end{tikzcd}$$
- identity 1-arrows: $\bullet = \bullet$
- identity axioms: $$\begin{tikzcd} \bullet \arrow[r, "f"] \arrow[rr, bend right, "\mathrm{id}\circ f=f"'] & \bullet \arrow[r, equal] & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r, equal] \arrow[rr, bend right, "f\circ \mathrm{id}=f"'] & \bullet \arrow[r, "f"] & \bullet \end{tikzcd}$$
- associativity axioms: $$\begin{tikzcd} & \bullet \arrow[dd, "g"] \arrow[dr, "h\circ g"] & \\ \bullet \arrow[ur, "f"] \arrow[rr, "h\circ(g\circ f)=(h\circ g)\circ f"'] \arrow[dr, "g\circ f"'] & & \bullet \\ & \bullet \arrow[ur, "h"'] & \end{tikzcd}$$

From 1-categories to $(\infty, 1)$-categories

The composition operation and associativity and unit axioms in a 1-category become higher data in an $(\infty, 1)$-category.

An $(\infty, 1)$-category has:

- objects $\bullet$; 1-arrows $\bullet \longrightarrow \bullet$; identity 1-arrows $\bullet = \bullet$
- composition
$$\begin{tikzcd} & \bullet \arrow[dr, "g"] \arrow[d, phantom, "\wr\|"] & \\ \bullet \arrow[ur, "f"] \arrow[rr, dashed, "g\circ f"'] & {} & \bullet \end{tikzcd}$$
witnessed by invertible 2-arrows
- identity composition witnesses
$$\begin{tikzcd} & \bullet \arrow[dr, equal] \arrow[d, phantom, "\wr\|"] & \\ \bullet \arrow[ur, "f"] \arrow[rr, "f"'] & {} & \bullet \end{tikzcd} \qquad \begin{tikzcd} & \bullet \arrow[dr, "f"] \arrow[d, phantom, "\wr\|"] & \\ \bullet \arrow[ur, equal] \arrow[rr, "f"'] & {} & \bullet \end{tikzcd}$$
- invertible 3-arrows witnessing associativity
$$\begin{tikzcd} & \bullet \arrow[dr, "h\circ g"] \arrow[dd, "g"] & \\ \bullet \arrow[ur, "f"] \arrow[rr, "h\circ g\circ f"] \arrow[dr, "g\circ f"'] & & \bullet \\ & \bullet \arrow[ur, "h"'] & \end{tikzcd}$$

A model for $(\infty, 1)$-categories

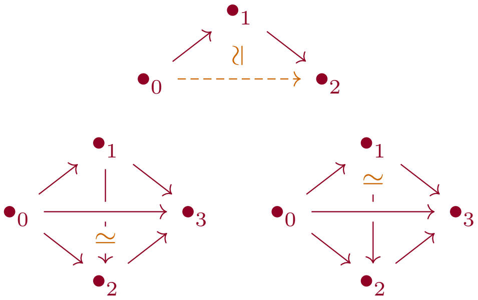
In a quasi-category, one popular model for an $(\infty, 1)$-category, this data is structured as a simplicial set with:

- 0-simplices = $\bullet$ = objects
- 1-simplices = $\bullet \longrightarrow \bullet$ = 1-arrows
- 2-simplices = $$\begin{tikzcd} & \bullet \arrow[dr, "g"] \arrow[d, phantom, "\wr|"] & \\ \bullet \arrow[ur, "f"] \arrow[rr, "h"'] & {} & \bullet \end{tikzcd}$$ = binary composites
- 3-simplices = $$\begin{tikzcd} & \bullet \arrow[dr, "k"] \arrow[dd, "g"] & \\ \bullet \arrow[ur, "f"] \arrow[rr, "\ell"] \arrow[dr, "j"'] & & \bullet \\ & \bullet \arrow[ur, "h"'] & \end{tikzcd}$$ = ternary composites
- n-simplices = n-ary composites
- with degenerate simplices used to encode identity arrows and identity composition witnesses

A model for $(\infty, 1)$-categories

A quasi-category is a "simplicial set with composition": a simplicial set in which every inner horn can be filled to a simplex.

Low dimensional horn filling:



An inner horn is the subcomplex of an n-simplex missing the top cell and the face opposite the vertex $\bullet_k$ for $0 < k < n$.

Corollary: In a quasi-category, all n-arrows with $n > 1$ are equivalences.

Summary: quasi-categories model ∞-categories

A quasi-category is a model of an infinite-dimensional category structured as a simplicial set.

- Basic data is given by low dimensional simplices:
  - 0-simplices = objects
  - 1-simplices = 1-arrows
- Axioms are witnessed by higher simplices:
  - 2-simplices witness binary composites
  - 3-simplices witness associativity of ternary composition
- Higher simplices also regarded as arrows: n-simplices = n-arrows
- Axioms imply that n-arrows are equivalences for $n > 1$.

Thus a quasi-category is an $(\infty, 1)$-category, with all n-arrows with $n > 1$ weakly invertible.

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Towards a simplicial model of $(\infty, 2)$-categories

Towards a simplicial model of an $(\infty, 2)$-category

How might a simplicial set model an $(\infty, 2)$-category?

- 0-simplices = $\bullet$ = objects
- 1-simplices = $\bullet \longrightarrow \bullet$ = 1-arrows
- 2-simplices = $\begin{tikzcd} & \bullet \arrow[dr, "g"] & \\ \bullet \arrow[ur, "f"] \arrow[rr, "h"'] \arrow[rr, phantom, shift left=3, "\Uparrow"] & & \bullet \end{tikzcd}$ = 2-arrows

Problem: the 2-simplices must play a dual role, in which they are

- interpreted as inhabited by possibly non-invertible 2-cells
- while also serving as witnesses for composition of 1-simplices

in which case it does not make sense to think of their inhabitants as non-invertible.

Idea: "mark" the 2-simplex witnesses for composition and demand that these marked 2-simplices behave like 2-dimensional equivalences.

Towards a simplicial model of an $(\infty, 2)$-category

- 2-simplices = $\begin{tikzcd} & \bullet \arrow[dr, "g"] \arrow[d, phantom, "\Uparrow"] & \\ \bullet \arrow[ur, "f"] \arrow[rr, "h"'] & {} & \bullet \end{tikzcd}$ = 2-arrows
- marked 2-simplices $\begin{tikzcd} & \bullet \arrow[dr, "g"] \arrow[d, phantom, "\wr"] & \\ \bullet \arrow[ur, "f"] \arrow[rr, "gf"'] & {} & \bullet \end{tikzcd}$ witness 1-arrow composition

Now 3-simplices witness composition of 2-arrows:

given $\begin{tikzcd} & \bullet_1 \arrow[r, "g"] \arrow[d, phantom, "\Uparrow\alpha"] & \bullet_2 \arrow[dr, "k"] \arrow[d, phantom, "\beta\Uparrow"] & \\ \bullet_0 \arrow[ur, "f"] \arrow[urr, "h"] \arrow[rrr, "\ell"'] & {} & {} & \bullet_3 \end{tikzcd}$ fill $\begin{tikzcd} & \bullet_2 \arrow[dr, "k"] \arrow[d, phantom, "\wr"] & \\ \bullet_1 \arrow[ur, "g"] \arrow[rr, dashed, "k\circ g"'] & {} & \bullet_3 \end{tikzcd}$ then fill

$$\begin{tikzcd} & \bullet_1 \arrow[r, "g"] \arrow[d, phantom, "\Uparrow\alpha"] & \bullet_2 \arrow[dr, "k"] \arrow[d, phantom, "\beta\Uparrow"] & \\ \bullet_0 \arrow[ur, "f"] \arrow[urr, "h"] \arrow[rrr, "\ell"'] & {} & {} & \bullet_3 \end{tikzcd} \simeq \begin{tikzcd} & \bullet_1 \arrow[r, "g"] \arrow[drr, "k\circ g"'] \arrow[d, phantom, "\Uparrow\alpha*\beta"] & \bullet_2 \arrow[dr, "k"] \arrow[d, phantom, "\wr"] & \\ \bullet_0 \arrow[ur, "f"] \arrow[rrr, "\ell"'] & {} & {} & \bullet_3 \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} & \bullet_1 \arrow[dr] \arrow[d, phantom, "\Uparrow"] & \\ \bullet_0 \arrow[ur] \arrow[rr] \arrow[dr] & {} \arrow[d, phantom, "\Downarrow"] & \bullet_3 \\ & \bullet_2 \arrow[ur] & \end{tikzcd}$$

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The complicial sets model of higher ∞-categories

Marked simplicial sets

For a simplicial set to model a higher ∞-category with non-invertible arrows in each dimension:

- It should have a distinguished set of "marked" n-simplices witnessing composition of $n-1$-simplices.
- Identity arrows, encoded by the degenerate simplices, should be marked.
- Marked simplices should behave like equivalences.
- In particular, 1-simplices that witness an equivalence between objects should also be marked.

This motivates the following definition:

> A marked simplicial set is a simplicial set with a designated subset of marked simplices that includes all degenerate simplices.

The symbol "$\simeq$" is used to decorate marked simplices.

Complicial sets

Recall:

A quasi-category is a "simplicial set with composition": a simplicial set in which every inner horn can be filled to a simplex.

A complicial set is a "marked simplicial set with composition": a simplicial set in which every admissible horn can be filled to a simplex and in which marked simplices satisfy the 2-of-3 property.

Low dimensional admissible horn filling:

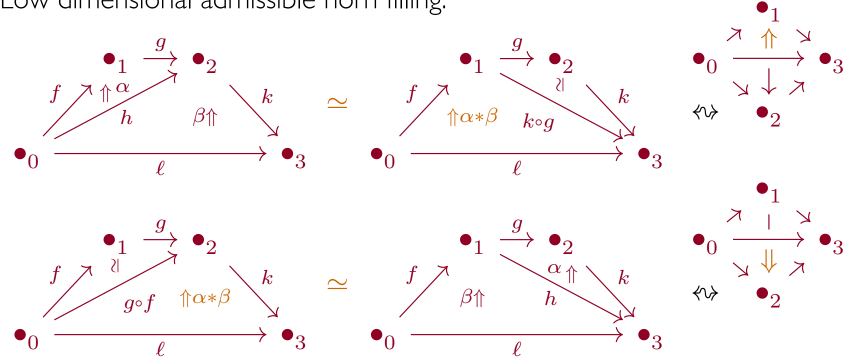
$$
\begin{tikzcd}
 & \bullet_1 \arrow[dr, "g"] \arrow[d, phantom, "\wr|"] & \\
\bullet_0 \arrow[ur, "f"] \arrow[rr, dashed, "g\circ f"'] & & \bullet_2
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \bullet_1 \arrow[dr, dashed, "f\circ e^{-1}"] \arrow[d, phantom, "\wr|"] & \\
\bullet_0 \arrow[ur, "e", "\sim"'] \arrow[rr, "f"'] & & \bullet_2
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \bullet_1 \arrow[dr, "e", "\sim"'] \arrow[d, phantom, "\wr|"] & \\
\bullet_0 \arrow[ur, dashed, "e^{-1}\circ f"] \arrow[rr, "f"'] & & \bullet_2
\end{tikzcd}
$$

and if f and g are marked so is $g \circ f$.

Complicial sets

A complicial set is a "marked simplicial set with composition": a simplicial set in which every admissible horn can be filled to a simplex and in which marked simplices satisfy the 2-of-3 property.

Low dimensional admissible horn filling:



and if α and β are marked so is $\alpha * \beta$.

Admissible horns

An n-simplex in a marked simplicial set is k-admissible — "its kth face is the composite of its $k-1$ and $k+1$-faces" — if every face that contains all of the vertices $\bullet_{k-1}, \bullet_k, \bullet_{k+1}$ is marked.

Marked faces include:

- the n-simplex
- all codimension-1 faces except the $(k-1)$th, kth, and $(k+1)$th
- the 2-simplex spanned by $\{\bullet_{k-1}, \bullet_k, \bullet_{k+1}\}$ when $0 < k < n$
- the edge spanned by $\{\bullet_0, \bullet_1\}$ when $k = 0$ or $\{\bullet_{n-1}, \bullet_n\}$ when $k = n$.

An k-admissible n-horn is the subcomplex of the k-admissible n-simplex that is missing the n-simplex and its k-th face.

Strict ω-categories as strict complicial sets

A strict complicial set is a complicial set in which every admissible horn can be filled uniquely, a "marked simplicial set with unique composition."

Any strict ω-category $\mathcal{C}$ defines a strict complicial set $N\mathcal{C}$, called the Street nerve, whose n-simplices are strict ω-functors

$$\mathcal{O}_n \to \mathcal{C},$$

where

- $\mathcal{O}_n$ is the free strict n-category generated by the n-simplex and
- an n-simplex is marked in $N\mathcal{C}$ just when the ω-functor $\mathcal{O}_n \to \mathcal{C}$ carries the top-dimensional n-arrow in $\mathcal{O}_n$ to an identity in $\mathcal{C}$.

Street-Roberts Conjecture (Verity). The Street nerve defines a fully faithful embedding of strict ω-categories into marked simplicial sets, and the essential image is the category of strict complicial sets.

Strict ω-categories as *weak* complicial sets

Strict ω-categories can also be a source of *weak* rather than *strict* complicial sets, simply by choosing a more expansive marking convention.

Any strict ω-category $\mathcal{C}$ defines a complicial set $N\mathcal{C}$ whose

- n-simplices are strict ω-functors $\mathcal{O}_n \to \mathcal{C}$ and where
- an n-simplex is marked in $N\mathcal{C}$ just when the ω-functor $\mathcal{O}_n \to \mathcal{C}$ carries the top-dimensional n-arrow in $\mathcal{O}_n$ to an equivalence in $\mathcal{C}$.

Moreover the complicial sets that arise in this way are saturated, meaning that every n-arrow equivalence is marked.

Saturation is a 2-of-6 property for marked simplices:

given an equivalence

$$\begin{tikzcd} & \bullet \arrow[dr, "\sim"] \arrow[dd, "f"] & \\ \bullet \arrow[ur, "g"] \arrow[dr, "\sim"'] \arrow[r, phantom, "\simeq"] & {} & \bullet \arrow[l, phantom, "\simeq"'] \\ & \bullet \arrow[ur, "h"'] & \end{tikzcd}$$

then f, g, h are also marked.

The n-complicial sets model of (∞, n)-categories

An n-complicial set is a saturated complicial set in which every simplex above dimension n is marked.

For example:

- the nerve of an ordinary 1-groupoid defines a 0-complicial set with everything marked
- the nerve of an ordinary 1-category defines a 1-complicial set with the isomorphisms marked
- the nerve of a strict 2-category defines a 2-complicial set with the 2-arrow isomorphisms and 1-arrow equivalences marked

In fact:

- A 0-complicial set is the same thing as a Kan complex, with everything marked.
- A 1-complicial set is exactly a quasi-category, with the equivalences marked.

Summary: complicial sets model higher ∞-categories

A complicial set is a model of an infinite-dimensional category structured as a marked simplicial set.

- Basic data is given by simplices:
  - 0-simplices = objects
  - n-simplices = n-arrows
- Axioms are witnessed by marked simplices:
  - marked n-simplices exhibit binary composites of $(n-1)$-simplices
- Marked simplices define invertible arrows:
  - marked n-simplices = n-equivalences
- In a saturated complicial set, all equivalences are marked.

An n-complicial set, a saturated complicial set in which every simplex above dimension n is marked, is a model of an (∞, n)-category.

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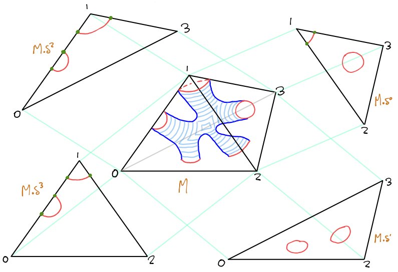
Complicial sets in the wild (joint with Dominic Verity)

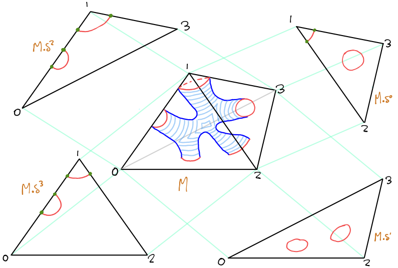
A simplicial set of simplicial bordisms (Verity)



A n-simplicial bordism is a functor from the category of faces of the n-simplex to the category of PL-manifolds and regular embeddings satisfying a boundary condition.

- Simplicial bordisms assemble into a semi simplicial set that admits fillers for all horns, constructed by gluing in cylinders.
- By a theorem of Rourke–Sanderson, degenerate simplices exist and make simplicial bordisms into a genuine Kan complex.

A complicial set of simplicial bordisms (Verity)



The Kan complex of simplicial bordisms can be marked in various ways:

- mark all bordisms as equivalences
- mark only trivial bordisms, which collapse onto their odd faces
- mark the simplicial bordisms that define h-cobordisms from their odd to their even faces

Theorem (Verity). All three marking conventions turn simplicial bordisms into a complicial set, and the third is the saturation of the second.

Complicial sets defined as homotopy coherent nerves

The homotopy coherent nerve converts a simplicially enriched category into a simplicial set.

Theorem (Cordier–Porter). The homotopy coherent nerve of a Kan complex enriched category is a quasi-category.

Theorem (Cordier–Porter). The homotopy coherent nerve of a 0-complicial set enriched category is a 1-complicial set.

Similarly:

Theorem*(Verity). The homotopy coherent nerve of a n-complicial set enriched category is a $n+1$-complicial set.

In particular, there are a plethora of 2-complicial sets of ∞-categories ...

The analytic vs synthetic theory of ∞-categories

The notion of an ∞-category is made rigorous by various models.

Q: How might you develop the category theory of ∞-categories?

Strategies:

- work analytically to give categorical definitions and prove theorems using the combinatorics of one model

  (eg., Joyal, Lurie, Gepner-Haugseng, Cisinski in $q\mathcal{C}at$; Kazhdan-Varshavsky, Rasekh in $\mathcal{R}ezk$; Simpson in $\mathcal{S}egal$)

- work synthetically to give categorical definitions and prove theorems in various models $q\mathcal{C}at$, $\mathcal{R}ezk$, $\mathcal{S}egal$, $1\text{-}\mathcal{C}omp$ at once

  (R-Verity: an ∞-cosmos axiomatizes the common features of the categories $q\mathcal{C}at$, $\mathcal{R}ezk$, $\mathcal{S}egal$, $1\text{-}\mathcal{C}omp$ of ∞-categories)

- work synthetically in a simplicial type theory augmenting homotopy type theory to prove theorems in $\mathcal{R}ezk$

  (R-Shulman: an ∞-category is a type with unique binary composites in which isomorphism is equivalent to identity)

∞-cosmoi of ∞-categories

Idea: an ∞-cosmos is a category in which ∞-categories live as objects that has enough structure to develop "formal category theory."

An ∞-cosmos is*:

- a quasi-categorically enriched category
- admitting "strict homotopy limits": flexible weighted simplicially enriched limits.

Examples of ∞-cosmoi:

- models of $(\infty, 1)$-categories: $q\mathcal{C}at$, $\mathcal{R}ezk$, $\mathcal{S}egal$, $1\text{-}\mathcal{C}omp$
- models of (∞, n)-categories: $n\text{-}q\mathcal{C}at$, $\theta_n\text{-}\mathcal{S}p$, $\mathcal{CSS}_n$, $n\text{-}\mathcal{C}omp$
- $\mathcal{C}at$, $\mathcal{K}an$, $\mathcal{C}omp$
- If $\mathcal{K}$ is an ∞-cosmos, so are $\mathcal{C}art(\mathcal{K})$, $co\mathcal{C}art(\mathcal{K})$ as well as the slices $\mathcal{K}_{/B}$, $\mathcal{C}art(\mathcal{K})_{/B}$, $co\mathcal{C}art(\mathcal{K})_{/B}$ over an ∞-category B.

Co/cartesian fibrations are on fleek

Challenge: define the Yoneda embedding as a functor between ∞-categories.

- Why is this so onerous? It's difficult to fully specify the data of a homotopy coherent diagram.
- Instead, an ∞-category-valued diagram can be repackaged as a co/cartesian fibration, with the homotopy coherence encoded by a universal property.

Idea: a co/cartesian fibration $E \overset{p}{\twoheadrightarrow} B$ is a family of ∞-categories E_b parametrized covariantly/contravariently by elements b of B.

The synthetic definition of a cocartesian fibration: a functor $E \overset{p}{\twoheadrightarrow} B$ so that $E^{\mathbb{2}} \overset{p}{\twoheadrightarrow} p{\downarrow}B$ admits a left adjoint right inverse.

The global universal property of co/cartesian fibrations

The codomain projection functor $\mathsf{cod}\colon \mathsf{co}\mathfrak{C}\mathsf{art}(\mathcal{K}) \to \mathcal{K}$ defines a "cartesian fibration of quasi-categorically enriched categories":

- For $F \overset{q}{\twoheadrightarrow} A$ and $E \overset{p}{\twoheadrightarrow} B$ in $\mathsf{co}\mathfrak{C}\mathsf{art}(\mathcal{K})$, the map $\mathsf{Fun}^{\mathsf{cart}}(q,p) \twoheadrightarrow \mathsf{Fun}(A,B)$ defines a cocartesian fibration in $\mathsf{q}\mathfrak{C}\mathsf{at}$.
- Pre- or post-composing by the arrows of $\mathsf{co}\mathfrak{C}\mathsf{art}(\mathcal{K})$ defines a cartesian functor between these cocartesian fibrations.
- A pullback $\begin{tikzcd} F \arrow[r, "g"] \arrow[d, two heads, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, two heads, "p"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$ forms a "cartesian lift of f with codomain p."

Consequently, for $U \hookrightarrow V$, cocartesian cocones with nadir p

$$\begin{tikzcd} \mathfrak{C}U^{\triangleright} \arrow[r] \arrow[d, hook] & \mathsf{co}\mathfrak{C}\mathsf{art}(\mathcal{K}) \arrow[d, "\mathsf{cod}"] \\ \mathfrak{C}V^{\triangleright} \arrow[r] \arrow[ur, dashed] & \mathcal{K} \end{tikzcd}$$

admit extensions that are unique up to a contractible space of choices.

The comprehension construction

A canonical lifting problem defines the comprehension construction:

$$\begin{tikzcd} & \mathfrak{C}\emptyset^{\triangleright} \arrow[r, "p\colon E \twoheadrightarrow B"] \arrow[d, hook] & \mathrm{co}\mathfrak{Cart}(\mathcal{K}) \arrow[r, two heads, "\mathrm{dom}"] \arrow[d, two heads, "\mathrm{cod}"] & \mathcal{K} \\ \mathfrak{C}B_0 \arrow[r, hook] & \mathfrak{C}B_0^{\triangleright} \arrow[r] \arrow[ur, dashed] & \mathcal{K} & \end{tikzcd}$$

which "straightens" p into a homotopy coherent diagram $c_p\colon \mathfrak{C}B_0 \to \mathcal{K}$ indexed by the underlying quasi-category of B.

Applying comprehension in $\mathcal{K}_{/A}$ to a universal fibration $\tilde{U} \overset{\pi}{\twoheadrightarrow} U$ in $\mathcal{K}$, yields $c_\pi\colon \mathfrak{C}\mathrm{Fun}(A,U) \to \mathrm{co}\mathfrak{Cart}(\mathcal{K})_{/A}$, which "unstraightens" an ∞-category-valued diagram into a cocartesian fibration over A.

Applying comprehension in $\mathfrak{Cart}(\mathcal{K})_{/A}$ to the cocartesian fibration $A^2 \overset{\mathrm{cod}}{\twoheadrightarrow} A$ constructs the Yoneda embedding $\mathfrak{C}A_0 \to \mathfrak{Cart}(\mathcal{K})_{/A}$.

References

For more on the complicial sets model of higher ∞-categories see:

Dominic Verity

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- Weak complicial sets I, basic homotopy theory, Adv. Math., 2008; arXiv:math/0604414
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Emily Riehl

- Complicial sets, an overture, 2016 MATRIX Annals, arXiv:1610.06801

Emily Riehl and Dominic Verity

- Elements of ∞-Category Theory, draft book in progress www.math.jhu.edu/~eriehl/elements.pdf (particularly Appendix D: the combinatorics of (marked) simplicial sets)