SKETCHES OF AN ELEPHANT: AN INTRODUCTION TO TOPOS THEORY

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Abstract. We briefly outline the history of topos theory, from its origins in sheaf theory which lead to the notion of a *Grothendieck topos*, through its unification with categorical logic which lead to the notion of an *elementary topos*, to a glimpse of the modern topos-theoretic outlook. P.T. Johnstone describes this point of view as "the rejection of the idea that there is a fixed universe of 'constant' sets within which mathematics can and should be developed, and the recognition that the notion of 'variable structure' may be more conveniently handled within a universe of continuously variable sets." Time permitting, we'll sketch an application of the universal language of the topos of sheaves on the spectrum of a commutative ring that allows one to regard the ring as a local ring, at least locally.

Contents

1. Sheaves of spaces

In calculus one studies continuous or continuously differentiable or k-times continuously differentiable functions $f\colon U \subset \mathbb{R}^n \to \mathbb{R}$ which might only be defined on certain (typically open) regions in Euclidean space: eg $f(x,y) = \frac{1}{x-y}$, which is a C^∞-function on the complement of the diagonal $x = y$ in the plane $\mathbb{R}^2$. This leads naturally to the consideration of the **sheaf of continuous real-valued functions of n-variables** whose data is given by a family of sets

$$C(U,\mathbb{R}) = \{\text{continuous functions } f\colon U \subset \mathbb{R}^n \to \mathbb{R}\}$$

for each open $U \subset \mathbb{R}^n$. What structure does this family of sets have?

- Firstly there are naturally defined **restriction functions**: whenever $V \subset U$ we have a function

$$\begin{array}{ccc} C(U,\mathbb{R}) & \xrightarrow{\text{res}^U_V} & C(V,\mathbb{R}) \\ f & \longmapsto & f|_V \end{array}$$

  and this assignment is functorial. Put concisely, the family $C(U,\mathbb{R})$ defines a contravariant functor

$$C(-,\mathbb{R})\colon \mathcal{O}^{\text{op}} \to \mathcal{S}\text{et}$$

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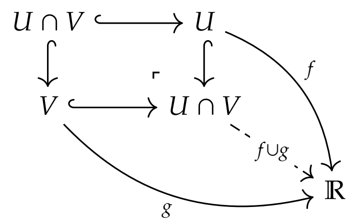
where $\mathcal{O}$ is the **poset of open subsets of** $\mathbb{R}^n$, with an object for each $U \subset \mathbb{R}^n$ and a morphism $V \to U$ iff $V \subset U$.

- Secondly there is a gluing property: whenever $f \in C(U,\mathbb{R})$ and $g \in C(V,\mathbb{R})$ restrict to the same function $f|_{U\cap V} = g|_{U\cap V} \in C(U\cap V,\mathbb{R})$, there is a unique continuous extension $f \cup g \in C(U\cup V,\mathbb{R})$ on account of the following pushout in the category of spaces and continuous functions:



This can be expressed more categorically by saying that the functor $C(-,\mathbb{R})$ carries the pushout defining $U\cup V$ by gluing U and V along $U\cap V$ to a pullback in the category of sets

$$\begin{array}{ccc} C(U\cup V,\mathbb{R}) & \xrightarrow{\mathrm{res}^{U\cup V}_{U}} & C(U,\mathbb{R}) \\ {\scriptstyle \mathrm{res}^{U\cup V}_{V}}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{res}^{U}_{U\cap V}} \\ C(V,\mathbb{R}) & \xrightarrow[\mathrm{res}^{V}_{U\cap V}]{} & C(U\cap V,\mathbb{R}) \end{array}$$

An analogous gluing property holds for any open cover $U = \cup_i U_i$.

This leads to the following definition.

defn. A **sheaf** (of sets) F on a space X is

- a **presheaf** $F\colon \mathcal{O}^{\mathrm{op}} \to \mathcal{S}\mathsf{et}$ indexed by the open set lattice of X
- so that for all open covers $U = \cup_i U_i$ the set $F(U)$ is the equalizer of the pair of functions

$$\textstyle\prod_i F(U_i) \underset{\mathrm{res}^{U_j}_{U_i\cap U_j}}{\overset{\mathrm{res}^{U_i}_{U_i\cap U_j}}{\rightrightarrows}} \prod_{i,j} F(U_i\cap U_j)$$

There are several important examples.

ex. Sheaves on a space X with open set lattice $\mathcal{O}$ include:

(i) For $0 \le k \le \infty$, there is a sheaf $U \mapsto C^k(U,\mathbb{R})$ of k-times continuously differentiable real-valued functions on X.

(ii) For any fixed space Y, there is a sheaf $U \mapsto C(U,Y)$ of continuous functions $U \to Y$.

(iii) There is a sheaf $U \mapsto \{V \in \mathcal{O} \mid V \subset U\}$.

(iv) For any fixed open subset $V \subset X$, there is a sheaf $\mathcal{O}(-,V)$ defined by

$$U \mapsto \mathcal{O}(U,V) = \begin{cases} * & \text{if } U \subset V \\ \emptyset & \text{if } U \not\subset V \end{cases}$$

(v) For any fixed continuous function $p\colon Y \to X$, there is a sheaf Γ_p defined by

$$U \mapsto \Gamma_p U = \left\{ \begin{array}{ccc} & & Y \\ & \overset{s}{\nearrow} & \downarrow{\scriptstyle p} \\ U & \hookrightarrow & X \end{array} \right\}$$

whose elements are **local sections**, continuous functions $s\colon U \to Y$ so that ps coincides with the inclusion of U into X.

non-ex. For a fixed set S, the constant functor $U \mapsto S$ defines a presheaf $\mathcal{O}^{\mathrm{op}} \to \mathcal{S}\mathsf{et}$ that is a sheaf if and only if S is the singleton set.

The following result reveals that all sheaves of spaces can be realized as sheaves of local sections of a suitable defined continuous function $p\colon Y \to X$ that has the special property of being **étale**, meaning that each $y \in Y$ has an open neighborhood on which p restricts to a homeomorphism. For any sheaf or indeed presheaf F and any point $x \in X$, the **stalk** of F at x is the filtered colimit $\mathsf{colim}_{U \ni x} F(U)$, indexed by the open neighborhoods $U \ni x$, of the sets $F(U)$. In particular each $s \in F(V)$, where $V \ni x$, represents an element of the stalk of F at x. When F is the sheaf of continuous real-valued functions, these elements, which are called **germs**, can be thought of as equivalence classes of continuous functions that agree in an infinitesimal neighborhood of the point x. The disjoint union $\coprod_{x \in X} \mathsf{colim}_{U \ni x} F(U)$ of the stalks is equipped with the finest topology that makes the maps

$$\begin{array}{ccc} U & \xrightarrow{\sigma_s^U} & \coprod_{x \in X} \mathsf{colim}_{V \ni x} F(V) \\ y & \longmapsto & (y, [s]) \end{array}$$

continuous for each $s \in F(U)$, and defines a space over X via the projection whose fibers are exactly the stalks.

For this result and others to follow, we include a modern reference where a detailed proof can be found, with apologies to the original discoverers.

Theorem ([J, 0.24])**.** *For any space X with poset of open subsets $\mathcal{O}$, there is an adjunction between the category of presheaves on $\mathcal{O}$ and the category of spaces over X*

$$\mathcal{S}\mathsf{et}^{\mathcal{O}^{\mathrm{op}}} \underset{\Gamma}{\overset{L}{\rightleftarrows}} \mathcal{S}\mathsf{pace}_{/X} \quad (L \dashv \Gamma)$$

that restricts to define an adjoint equivalence between the category of sheaves on X and the category of étale spaces over X.

$$\mathcal{S}\mathsf{hv}(X) \underset{\Gamma,\ \sim}{\overset{L,\ \sim}{\rightleftarrows}} \mathcal{E}\mathsf{tale}_{/X} \quad (L \dashv \Gamma)$$

Proof. The right adjoint Γ takes a space $p\colon Y \to X$ over X to the sheaf of local sections over X. The left adjoint L takes a presheaf F to the étale mapping

$$\pi\colon \coprod_{x \in X} \mathsf{colim}_{U \ni x} F(U) \to X$$

whose fiber over x is the stalk of F. Note the topology of the space of stalks is arranged to guarantee that there is a canonical local section σ_s^U associated to each $s \in F(U)$. This defines the components of the unit natural transformation $\eta\colon F \to \Gamma L(F)$ for any $F \in \mathcal{S}\mathsf{et}^{\mathcal{O}^{\mathrm{op}}}$. Conversely, for any $p\colon Y \to X$ there is a canonical mapping $\epsilon\colon L\Gamma_p \to Y$ over X defined by choosing a representative $s \in \Gamma_p(U)$ for each germ of x and evaluating to obtain a point $s(x) \in Y$ in the fiber over x. This is well-defined and gives the components of the counit natural transformation. ◻

The colimits that define the stalks are **filtered colimits**, meaning that for each pair of open neighborhoods $U, V \ni x$ there is a common refinement $U \cap V \ni x$. It follows that the left adjoint L preserve finite limits. Hence:

Corollary ([J, 0.25]). *The category of sheaves on X defines a lex reflective subcategory of the category of presheaves, which is to say that the inclusion admits a finite-limit-preserving left adjoint*

$$\mathcal{S}\mathsf{hv}(X) \underset{\hookrightarrow}{\overset{\Gamma L}{\underset{\perp}{\dashleftarrow}}} \mathcal{S}\mathsf{et}^{\mathcal{O}^{\mathrm{op}}}$$

called the ***associated sheaf functor****.*

Proof. The claimed adjunction is defined as the composite of the adjunction $L \dashv \Gamma$ with the adjoint equivalence

$$\mathcal{S}\mathsf{hv}(X) \underset{\underset{L}{\sim}}{\overset{\overset{\Gamma}{\sim}}{\underset{\perp}{\rightleftarrows}}} \mathcal{E}\mathsf{tale}_{/X} \underset{\Gamma}{\overset{L}{\underset{\perp}{\rightleftarrows}}} \mathcal{S}\mathsf{et}^{\mathcal{O}^{\mathrm{op}}}$$ □

Exercise. What does it mean for $F\colon \mathcal{O}^{\mathrm{op}} \to \mathcal{S}\mathsf{et}$ to be a sheaf if X is discrete or indiscrete? What does the sheaf condition mean when X is your favorite finite topological space?

2. Generalizations

The characterization of sheaves as sheaves of continuous sections of the canonical projection from the space of stalks is special to sheaves of spaces, and has no analog for the more general notions of sheaves. However, the characterization of the *category* of sheaves as a lex reflective subcategory of the category of presheaves is prototypical and in fact characterizes the Grothendieck topoi, which are typically defined in a different manner.

defn. A **Grothendieck topos** $\mathcal{S}\mathsf{hv}(\mathcal{C}, \tau)$ is a lex reflective subcategory of the category of presheaves indexed by an arbitrary small category $\mathcal{C}$

$$\mathcal{S}\mathsf{hv}(\mathcal{C}, \tau) \underset{\hookrightarrow}{\overset{a}{\underset{\perp}{\dashleftarrow}}} \mathcal{S}\mathsf{et}^{\mathcal{C}^{\mathrm{op}}}$$

As the notation suggests, for each Grothendieck topos $\mathcal{S}\mathsf{hv}(\mathcal{C}, \tau)$ there is a way to characterize which presheaves $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathsf{et}$ define sheaves. The τ in this notation refers to something called a **Grothendieck topology**, which is analogous to the topology encoded by the open set lattice $\mathcal{O}$ associated to a topological space X. We decline to give the full definition here but describe the main idea.

Objects $U \in \mathcal{C}$ can be thought of as analogous to the "open sets" that parametrize a sheaf on a space. As before, a presheaf $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathsf{et}$ comes with specified "restriction functions" $Ff\colon FU \to FV$ associated to each arrow $f\colon V \to U$ in $\mathcal{C}$. To state the condition that tells us which presheaves are sheaves, we need to specify which families of arrows with codomain U define "open covers" of U, playing the role of the family of inclusions $U_i \hookrightarrow U$ in an open cover $U = \cup_i U_i$ in $\mathcal{O}$. The main part of what is entailed in a Grothendieck topology τ is the specification of a collection of **covering sieves** for each $U \in \mathcal{C}$, these being subfunctors $S \hookrightarrow \mathcal{C}(-, U)$ of the representable functor determined by U. Then

defn. A presheaf $F\colon \mathcal{C}^{\mathrm{op}} \to \mathit{Set}$ is a **sheaf** if for every covering sieve $S \hookrightarrow \mathcal{C}(-, U)$ of each $U \in \mathcal{C}$ any map of presheaves admits a unique extension:

$$\begin{array}{ccc} S & \longrightarrow & F \\ \big\downarrow & \nearrow_{\exists!} & \\ \mathcal{C}(-, U) & & \end{array}$$

ex. In the **minimial Grothendieck topology**, the only covering sieves are the maximal covering sieves, containing all arrows with fixed codomain. These correspond to the maximal subobject $\mathcal{C}(-, U) = \mathcal{C}(-, U)$. Hence every presheaf is a sheaf. In particular, for any $\mathcal{C}$, the category of presheaves $\mathcal{S}\mathsf{et}^{\mathcal{C}^{\mathrm{op}}}$ is a Grothendieck topos.

See "sheaf toposes are equivalently the left exact localizations of presheaf toposes" on the nLab for more.

Exercise. Convert the definition of a sheaf just given to one that expresses the set $F(U)$ as a limit.

3. Elementary topoi

In the mid 1960s, Bill Lawvere introduced the elementary theory of the category of sets in a paper by that same name, providing a categorical characterization of $\mathcal{S}\mathsf{et}$ as a category. The notion of an elementary topos grew out of a collaboration with Myles Tierney, culminating in an axiomatization that captured many of the logical properties of the category of sets via axioms that are also satisfied in any Grothendieck topos.

defn. An **elementary topos** is a category $\mathcal{E}$ that

- has finite limits,
- is cartesian closed, and
- has a **subobject classifier**, this being an object $\Omega \in \mathcal{E}$ equipped with a distinguished monomorphism $\top\colon 1 \rightarrowtail \mathcal{E}$ whose domain is necessarily terminal with the property that arrows $A \to \Omega$ correspond bijectively to subobjects[1] of A, constructed by forming the pullback

$$\begin{array}{ccc} S & \longrightarrow & 1 \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \top} \\ A & \longrightarrow & \Omega \end{array}$$

ex. The prototypical example is the category of sets, with $\Omega = \{\top, \bot\}$ classifying subsets $S \subset A$ via their characteristic function

$$A \xrightarrow{\chi_S} \Omega$$
$$a \longmapsto \begin{cases} \top & \text{if } a \in S \\ \bot & \text{if } a \notin S \end{cases}$$

Importantly, every Grothendieck topos is an elementary topos.

Theorem. *$\mathcal{S}\mathsf{et}^{\mathcal{C}^{\mathrm{op}}}$, and $\mathcal{S}\mathsf{hv}(\mathcal{C}, \tau)$ are elementary topoi.*

[1]A **subobject** of A is a monomorphism $S \rightarrowtail A$ up to isomorphism: if S and S' are isomorphic over A, they represent the same subobject.

Proof. Famously the category $\mathcal{S}\mathsf{et}^{\mathcal{C}^{\mathrm{op}}}$ is cartesian closed with $G^F(U)$ defined to be the set of natural transformations $F \times \mathcal{C}(-, U) \to G$. This category is also complete and cocomplete. By the Yoneda lemma, the subobject classifier $\Omega \in \mathcal{S}\mathsf{et}^{\mathcal{C}^{\mathrm{op}}}$ must be defined so that $\Omega(U)$ is the set of subobjects of $\mathcal{C}(-, U)$. Thus, we define $\Omega(U)$ to be the set of sieves on U.

As a lex reflective subcategory of $\mathcal{S}\mathsf{et}^{\mathcal{C}^{\mathrm{op}}}$, $\mathcal{S}\mathsf{hv}(\mathcal{C}, \tau)$ is complete and cocomplete, with limits formed as in $\mathcal{S}\mathsf{et}^{\mathcal{C}^{\mathrm{op}}}$ and colimits formed by applying the associated sheaf functor to the colimit of presheaves. The exponential G^F for sheaves G and F is defined as in $\mathcal{S}\mathsf{et}^{\mathcal{C}^{\mathrm{op}}}$. It follows from the fact that a preserves products that this is automatically a sheaf. By adjunction and the Yoneda lemma, the subobject classifier is defined by taking $\Omega(U)$ to be the set of subobjects of the associated sheaf of $\mathcal{C}(-, U)$. Further details can be found in [MM]. □

This proof reveals that all Grothendieck topoi have all limits and colimits, not merely finite ones. Thus, we have the following example of an elementary topos that is not a Grothendieck topos.

ex. The category of finite sets is an elementary topos, with finite limits, exponentials, and subobject classifier inherited from the topos of sets.

The original definition of elementary topos included an axiom requiring the existence of finite colimits. Christian Mikkelsen later noticed that this is unnecessary, and Bob Paré found a particularly elegant justification for the redundancy:

Theorem ([MM, §IV.5]). *For any elementary topos $\mathcal{E}$, the functor $\Omega^{-} \colon \mathcal{E}^{\mathrm{op}} \to \mathcal{E}$ is monadic. Consequently, any elementary topos has all finite colimits, created by the contravariant power object functor.*

Not only are elementary topoi cartesian closed, but they are **locally cartesian closed**, meaning that for each $f \colon A \to B$ in $\mathcal{E}$, the pullback functor admits left and right adjoints

$$\mathcal{E}_{/B} \xrightarrow{f^*} \mathcal{E}_{/A}, \qquad \Sigma_f \dashv f^* \dashv \Pi_f \colon \mathcal{E}_{/A} \to \mathcal{E}_{/B}$$

This follows from a result sometimes referred to as the "fundamental theorem of elementary topos theory" and the fact that locally cartesian closed categories are equivalently characterized as cartesian closed categories whose slice categories $\mathcal{E}_{/A}$ are again cartesian closed.

Theorem ([MM, IV.7.1]). *For any elementary topos $\mathcal{E}$ and any object $A \in \mathcal{E}$, $\mathcal{E}_{/A}$ is an elementary topos.*

We advertised elementary topoi as categories that have many of the logical properties of the category of sets. To see this note that we can form the following constructions in any elementary topos:

- Each object $A \in \mathcal{E}$ has a **power object** Ω^A whose elements $1 \to \Omega^A$ classify subobjects of A.
- The **membership relation** is encoded by the evaluation map $\epsilon_A \colon A \times \Omega^A \to \Omega$, arising from the cartesian closed structure on $\mathcal{E}$.
- The **identity relation** is encoded by the map $=_A \colon A \times A \to \Omega$ that classifies the diagonal subobject $\Delta_A \colon A \to A \times A$.

Moreover, in an elementary topos the set of subobjects of a fixed object A naturally admits the structure of a **Heyting algebra**, a cartesian closed bicomplete poset. Consequently:

Theorem. *The subobject classifier in an elementary topos admits the structure of a* ***internal Heyting algebra****, admitting maps*

$$1 \underset{\bot}{\overset{\top}{\rightrightarrows}} \Omega \circlearrowleft^{\neg} \quad \Omega \xleftarrow{\wedge} \Omega \times \Omega, \quad \Omega \xleftarrow{\vee} \Omega \times \Omega, \quad \Omega \xleftarrow{\Rightarrow} \Omega \times \Omega$$

satisfying axioms that can be expressed internally to any category with finite limits.

4. The internal language

The idea of the logic encoded by an elementary topos can be made more precise. The Mitchell-Bénabou language has types and terms, used to build formulae, which can be interpreted in the topos. We briefly sketch its definition.

defn. In the Mitchell-Bénabou language:

- Types A, B, C, correspond to objects of $\mathcal{E}$.
- Terms, which will be represented by arrows of $\mathcal{E}$ and belong to the type corresponding to their codomain, are built inductively from:
  - variables a, a', a'' of type A, which are interpreted by the identity map;
  - by pairing terms $u : A$ and $v : B$, represented by maps $u \colon U \to A$ and $v \colon V \to B$, to form a term $\langle u, v\rangle : A \times B$, which is represented by the map $u \times v \colon U \times V \to A \times B$;
  - by composing a function $f \colon A \to B$ with a term $u : A$ to obtain a new term $f \circ u : B$ represented by the composite function;
  - by applying a term $\phi : B^A$ to a term $u : A$ to obtain a term $\phi(u) : B$ represented by the composite function
$$U \times V \xrightarrow{u \times \phi} A \times B^A \xrightarrow{\epsilon} B$$
  - in the special case, by applying a term $\tau : \Omega^A$ to a term $u : A$ to obtain a term $u \in \tau : \Omega$ represented by the composite function
$$U \times W \xrightarrow{u \times \tau} A \times \Omega^A \xrightarrow{\epsilon} \Omega$$
  - by combining a variable $a : A$ with a term represented by a function $\sigma \colon A \times B \to C$ to obtain a term $\lambda a.\sigma : C^A$ represented by the function $\sigma \colon B \to C^A$.

Terms of type Ω are called **formulae**. For instance, given two terms $u, v : A$, the composite morphism

$$U \times V \xrightarrow{u \times v} U \times V \xrightarrow{=_A} \Omega$$

defines a formula $u =_A v$. Formulae can be combined using the operations $\wedge$, $\vee$, $\Rightarrow$, and $\neg$ to form combined propositional formulae. In addition, any elementary topos has "internal adjoints" to the constant arrow

$$\Omega \xrightarrow{\Delta} \Omega^A, \qquad \forall, \exists \colon \Omega^A \to \Omega$$

that can be used to quantify formulae involving a free variable of type A.

5. All rings are locally local rings

Let R be a commutative, unital ring and consider its set $\mathrm{Spec}(R)$ of prime ideals, equipped with the **Zariski topology**, whose basic opens are the sets $O_r = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid r \notin \mathfrak{p}\}$ indexed by elements $r \in R$. By the first theorem above, we can define a sheaf on $\mathrm{Spec}(R)$ by defining an étale space over $\mathrm{Spec}(R)$. We build this from the local rings

$$R_{\mathfrak{p}} = \{\frac{r}{s} \mid r, s \in R, s \notin \mathfrak{p}\}.$$

These are **local rings**: satisfying the condition that for each $x \in R_{\mathfrak{p}}$ either x or $1-x$ is invertible. We equip the disjoint union $\coprod_{\mathfrak{p}\in\mathrm{Spec}(R)} R_{\mathfrak{p}}$ with the finest topology that makes the maps

$$\begin{array}{ccc} O_s & \xrightarrow{\sigma^r_s} & \coprod_{\mathfrak{q}\in\mathrm{Spec}(R)} R_{\mathfrak{q}} \\ \mathfrak{p} & \longmapsto & (\mathfrak{p}, \frac{r}{s}) \end{array}$$

continuous for all $r, s \in R$. This defines the **structure space** of the ring R.

Theorem ([B, 2.11.8,9,15]). *The structure space defines an étale space*

$$\pi \colon \coprod_{\mathfrak{p}\in\mathrm{Spec}(R)} R_{\mathfrak{p}} \to \mathrm{Spec}(R)$$

whose corresponding sheaf of local sections Γ *is a sheaf of rings. Furthermore, the ring of global sections* $\Gamma(\mathrm{Spec}(R))$ *is isomorphic to* R.

A sheaf of rings is a sheaf valued in the category $\mathcal{R}\mathsf{ing}$ of rings rather than the category $\mathcal{S}\mathsf{et}$ of sets. This allows us to think about Γ as a ring object in the category of sheaves of sets on $\mathrm{Spec}(R)$. The following result explains the sense in which the "ring" Γ is "locally a local ring."

Proposition ([B, 2.11.16]). *Let* R *be a commutative ring with unit and* Γ *the sheaf of local sections of the corresponding structure space. Then for every open* $U \subset \mathrm{Spec}(R)$ *and* $\sigma \in \Gamma(U)$, *there exists an open cover* $U = \cup_i U_i$ *such that for each* i *either* $\sigma|_{U_i}$ *is invertible in* $\Gamma(U_i)$ *or* $(1-\sigma)|_{U_i}$ *is invertible in* $\Gamma(U_i)$.

The Kripke-Joyal semantics tells us how to determine the validity of a formula φ written in the internal language of a topos. In the topos of sheaves on the space $\mathrm{Spec}(R)$ we have a formula

$$\forall x : \Gamma, (\exists u : \Gamma, x \cdot u = 1) \vee (\exists v : \Gamma, (1-x) \cdot v = 1).$$

Each $r \in R$ defines a global section $r \in \Gamma(\mathrm{Spec}(R))$, so the previous result tells us there exists an open cover $\cup_i U_i$ of $\mathrm{Spec}(R)$ so that for each U_i either $r|_{U_i}$ or $(1-r)|_{U_i}$ is invertible. In the Kripke-Joyal semantics, this tells us that this formula is then valid. This justifies treating the commutative ring R as if it were "locally a local ring" when proving theorems in the internal logic of the topos of sheaves on the space $\mathrm{Spec}(R)$.

References

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