INDUCTION AND CONSTRUCTION: THE POINTLESS THEORY OF LOCALIC TOPOX

tslil clingman

Abstract. In this talk we will explore a half-way notion between the concrete description of the topos of sheaves of continuous functions on a topological space, and the fully general description of topos on a site. Specifically, we will concern ourselves with the replacement of topological spaces by categories of a particular nature. These categories are the locales of the title, and we will sketch some of their theory that is of independent interest so as to gain an intuition for the weirdness to come. To mention some highlights, we will encounter spaces without points, and examine both extrinsic and *intrinsic* descriptions of when a topos is equivalent to a topos of sheaves on a locale. The connection with topological spaces will then motivate the definition of geometric morphisms, whose properties and utility will be of central concern in the coming talks. For those playing along at home, references include, as ever, Mac Lane – Moerdijk [MM] (the chapter bearing the same name as the title of this talk), the nLab (articles: locale, localic topos, and geometric morphism) but also the book of Picado – Pultr, "Frames and Locales" [PP].

Contents

For a space X with a topology τ we considered the lattice of open sets ΩX of X determined by the topology and the inclusion relation $\subseteq$. The objects of ΩX are the open subsets of X and there is a unique morphism from U to V just when $U \subset V$. Then for any other space Y, such as $\mathbb{R}$, we can define a contravariant functor

$$C(-, Y) \colon \Omega X^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$$

by defining $C(U, Y)$ to be the set of continuous functions from U to Y. This presheaf satisfies a "local to global" condition we referred to as the *sheaf condition*, defining a sheaf on X. This leads to the following question:

Q. What are the open aspects of X we need to be able to form the category of sheaves on X?

Are the open sets enough?

1. Answer

We'll follow the following outline:

(i) We'll start with a topological space (X, τ).
(ii) We'll generalize this to a lattice $(Y, \subset)$.

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(iii) We'll generalize this to a suitably structured category.
(iv) We'll conclude by constructing sheaves on this.

defn. If $(L, \leq)$ is a poset (meaning the relation is anti-symmetric, transitive, and reflexive) and we consider a family of elements $\{x_i\}_{i \in I} \subset L$.

(i) The **meet** $\wedge x_i$, when it exists, is the largest $x \in L$ such that $x \leq x_i$ for all $i \in I$.
(ii) The **join** $\vee x_i$, when it exists, is the smallest $x \in L$ such that $x_i \leq x$ for all $i \in I$.

We say that meets and joints are finite if the indexing set I is finite.

Lemma-Exercise.

(i) In the lattice of opens $(\tau, \subset)$ associated to a topological space (X, τ) joins are unions and always exist.
(ii) If $(L, \leq)$ has all joins then it has all meets!
(iii) $x \wedge (\vee y_i) \geq \vee(x \wedge y_i)$
*(iv) Upon considering $(L, \leq)$ as a **thin category** — that is a category with at most one arrow in each hom-set — $\prod_{i \in I} L(x_i, z) \cong L(z, \wedge x_i)$*

Exercise. Consequently, the lattice of opens $(\tau, \subset)$ associated to a topological space (X, τ) has all meets. We leave it as an exercise to work out what they are. Consequently ΩX is a complete and cocomplete category (cccc).

Remark. Of course properties (ii), (iii), (iv) apply to the lattice $(L, \leq^{\mathrm{op}})$ and thus dualize to give us two more theorems for free about $(L, \leq)$.

A further exercise is to prove all of this diagrammatically.

NB. Note the lattice $(L, \leq^{\mathrm{op}})$ is L^{op} as a category.

NB. Note that $\Omega X = (\tau, \subseteq) \cong \mathsf{Top}(X, \mathbb{S})$ as categories, where $\mathbb{S}$ is the Sierpinski space, with two points, one of which is open.

In ΩX, the property (iii) above becomes an equality:

Lemma-Exercise. *In ΩX, $x \wedge (\vee y_i) = \vee(x \wedge y_i)$.*

NB. There exist spaces X so that in ΩX, $x \vee (\wedge y_i) \neq \wedge(x \vee y_i)$.

defn. A **frame** is a thin complete and cocomplete category which satisfies the infinite distributive law $x \wedge (\vee y_i) = \vee(x \wedge y_i)$.

Remark. The infinite distributive law $x \wedge (\vee y_i) = \vee(x \wedge y_i)$ is a decategorification of one of Giraud's axioms for Grothendieck topos: the axiom that says that colimits are universal.

ex. Every Heyting algebra is a frame.

ex. For any topological space (X, τ), ΩX is a frame.

Theorem-Exercise. *A poset P is a frame if and only if the Yoneda embedding*

$$\begin{tikzcd} P \arrow[r, hook, "よ"'] \arrow[r, phantom, shift left=2, "\perp"] & \mathbb{2}^{P^{\mathrm{op}}} \arrow[l, dashed, bend right] \end{tikzcd}$$

has a left exact left adjoint.

This is like the characterization of Grothendieck topox as lex reflexive subcategories of presheaf categories but it's better because the frame P is given as a lex reflexive subcategory of boolean-valued presheaves on itself, with the embedding given by the Yoneda embedding.

Q. Given a continuous function $f\colon (X,\tau) \to (Y,\sigma)$, what does this mean for their frames ΩX and ΩY?

Lemma-Exercise. *The inverse image function* $f^{\leftharpoondown} : \mathfrak{P}Y \to \mathfrak{P}X$

(i) descends to a function $\Omega Y \to \Omega X$
(ii) which is cocontinuous (preserves joins)
(iii) and preserves finite meets (is lex)

These properties motivate the following definition:

defn. Given frames F and F' a **frame morphism** $f\colon F \to F'$ is a lex cocontinuous functor, a finite meet and join preserving order-preserving map.

Corollary. $\Omega\colon \mathcal{T}\mathsf{op} \to \mathcal{F}\mathsf{rm}^{\mathsf{op}}$ *is a functor.*

The contravariance is annoying, so we'd like to drop the op.

Q (op-drop warmup)**.** Given a frame F and $x \in F$, does the functor $- \wedge x\colon F \to F$ have a right adjoint?

If there were a right adjoint $x \Rightarrow -\colon F \to F$, we would need to have

$$F(y \le x, z) \cong F(y, x \Rightarrow z)$$

for all y and z. Recall that $F(y \le x, z) \subset \{*\}$ so if $y \le x \le z$ then we would have to have $y \le x \Rightarrow z$. So we could try defining $x \Rightarrow z$ to be the join

$$x \Rightarrow z := \vee\{w \in F \mid w \wedge x \le z\}.$$

Indeed then $y \le \vee\{w \in F \mid w \wedge x \le z\}$ if and only if $\exists w$ so that $y \le w$ and $w \wedge x \le z$ which is the case if and only if $y \wedge x \le z$.

Corollary. *A frame is a complete and cocomplete cartesian closed category (cccccc).*

Exercise. What is $\Rightarrow$ in ΩX?

What does this have to do with dropping the op in the functor $\Omega\colon \mathcal{T}\mathsf{op} \to \mathcal{F}\mathsf{rm}^{\mathsf{op}}$?

Wouldn't it be great that if for any morphism of frames $f\colon F' \to F$ we could magically extract an arrow pointing in the other way. As it turns out every frame morphism, considered as a functor between thin categories, has a right adjoint $g\colon F \to F'$.

We can work out the definition as follows. The defining universal property says that

$$F(f(y), z) \cong F'(y, g(z)),$$

i.e., $f(y) \le z$ iff $y \le g(z)$. We can use the same trick and define

$$g(z) := \vee\{w \in F' \mid f(w) \le z\}.$$

Remark. This is the usual formula provided by the adjoint functor theorem

$$g(z) := \mathsf{colim}(f/z \to F').$$

Be careful:

Warning. g is not in general a frame morphism!

In other words, the process of moving from f to its right adjoint g does not define an identity-on-objects contravariant involution $\mathcal{F}\mathsf{rm}^{\mathsf{op}} \to \mathcal{F}\mathsf{rm}$, since the arrows g are not morphisms of frames. Instead we land in the following category:

defn. $\mathcal{L}\mathsf{oc}$ is the category whose objects are frames and whose morphisms $f\colon L \to L'$ are continuous functors with chosen left exact left adjoints (meaning the left adjoints are frame morphisms).

The intent of this definition is that we have a contravariant equivalence of categories:

Exercise. $\mathcal{F}\mathsf{rm}^{\mathsf{op}} \simeq \mathcal{L}\mathsf{oc}$.

Note the morphisms in the category of locales go in the direction of a continuous function: given $f\colon (X,\tau) \to (Y,\sigma)$, there is a locale morphism $f\colon \Omega X \to \Omega Y$.

Corollary. *There is a functor* $\mathrm{Lc}\colon \mathcal{T}\mathsf{op} \to \mathcal{L}\mathsf{oc}$ *defined on objects by* Ω.

Even though the objects of the categories $\mathcal{F}\mathsf{rm}$ and $\mathcal{L}\mathsf{oc}$ are the same they should be thought of differently. The objects of $\mathcal{F}\mathsf{rm}$ should be thought of as logical where the objects of $\mathcal{L}\mathsf{oc}$ should be thought of as topological.

After the break we'll define points of locales (objects of $\mathcal{L}\mathsf{oc}$) and look for exotic examples of frames, other than the ones (spaces, Heyting algebras) mentioned above.

2. After the break

We started with the category $\mathcal{F}\mathsf{rm}$, whose objects are frames (complete and cocomplete categories that satisfy the infinite distributive law) and whose morphisms are finite meet preserving and join preserving order-preserving maps. We discovered that any morphism in the category of frames has a right adjoint, which doesn't preserve joins but does preserve arbitrary meets (since right adjoints preserve limits). Thus we define a new category $\mathcal{L}\mathsf{oc}$ whose objects again are frames but whose morphisms are meet-preserving functors that have a left adjoint that preserves finite meets (as well as arbitrary joins, since left adjoints preserve colimits).

Lemma-Exercise. *If X and Y are Hausdorff spaces, then*

$$\mathcal{F}\mathsf{rm}(\Omega Y, \Omega X) \cong \mathcal{T}\mathsf{op}(X, Y).$$

That is the only morphisms of frames between open set lattices of Hausdorff spaces arise from continuous functions. Note that the points of a space X are recovered by continuous functions $1 \to X$, i.e., $X \cong \mathcal{T}\mathsf{op}(1, X)$. Thus, if X is a Hausdorff space

$$X \cong \mathcal{T}\mathsf{op}(1, X) \cong \mathcal{F}\mathsf{rm}(\Omega X, \Omega 1) \cong \mathcal{L}\mathsf{oc}(\Omega 1, \Omega X).$$

Note $\Omega 1$ is the frame $\varnothing \leq *$. So we generalize this to define the "points" of any locale. Motivated by this we write $\mathbb{P} := \Omega 1$.

defn. Given a locale L, a **point** in L is a morphism of locales $\mathbb{P} \to L$.

Lemma-Exercise. *A point in L is equivalently*

(i) a frame morphism $L \to \mathbb{P}$
(ii) a ***completely prime filter***
(iii) a ***meet irreducible element*** *of L*

Warning. The equivalence between (ii) and (iii) is classical, rather than constructive. The second characterization is better.

defn. A **filter** $\mathfrak{F}$ on a poset Q is an

- upwards closed subset: meaning $x \in \mathfrak{F}$ and $y \geq x$ implies $y \in \mathfrak{F}$ that is
- that's down directed: $x, y \in \mathfrak{F}$, then there exists $z \in \mathfrak{F}$ so that $z \leq x \wedge y$

A **completely prime** filter satisfies the additional property that

- $\vee x_i \in \mathfrak{F}$ implies that there exists i so that $x_i \in \mathfrak{F}$.

defn. An element $p \neq \top \in L$ is **meet-irreducible** if whenever $x \wedge y \leq p$ then $x \leq p$ or $y \leq p$.

ex. There are lots of locales without any points, whose open set lattices have a top element, a bottom element, and an arbitrary set of at least three uncomparable elements in between.

$$\begin{tikzcd} & \top & \\ \bullet \arrow[ur, dash] & \bullet \arrow[u, dash] & \bullet \arrow[ul, dash] \\ & \bot \arrow[ul, dash] \arrow[u, dash] \arrow[ur, dash] & \end{tikzcd}$$

These are distinct locales none of which have any points. So points don't determine a locale!

There's a lot of information that locales understand that point set topology does not.

Theorem. *There is a functor* $\mathrm{Sp}\colon \mathcal{L}\mathsf{oc} \to \mathcal{T}\mathsf{op}$ *that carries a locale to a space whose underlying set is the set of its points. This is right adjoint to the functor* Lc.

defn. A **sublocale** is a regular subobject $L' \rightarrowtail L$ in $\mathcal{L}\mathsf{oc}$ (a regular quotient in $\mathcal{F}\mathsf{rm}$).

These are much richer than subspaces in topology.

Theorem. *A sublocale on L is equivalently a map $\nu\colon L \to L$ (not of locales or frames, just a map) satisfying $x \leq \nu x$, $x \leq y \implies \nu x \leq \nu y$, $\nu^2 = \nu$, and $\nu(x \wedge y) = \nu x \wedge \nu y$.*

The sublocale is then the equalizer of ν and the identity, the fixed points for ν. This is reminiscent of Lawvere-Tierney operators.

We can define what it means for a sublocale to be **dense** in such a way that dense subspaces become dense sublocales.

Theorem. *There is a smallest dense sublocale of any locale.*

In particular, if D and D' are dense sublocales, then $D \cap D'$ is dense. In particular $\mathbb{Q} \cap \mathbb{R} \backslash \mathbb{Q}$ is dense in $\mathbb{R}$! It has no points, but it's dense. Again, the points are not the point.

This can resolve the Banach-Tarski paradox. The points are disjoint as subspaces but not as sublocales. You didn't partition the points of the solid ball to get two solid balls. You duplicated information. See Alex Simpson "Measure, randomness, and sublocales."

defn. Let L be a locale. A functor $P\colon L^{\mathrm{op}} \to \mathcal{S}\mathsf{et}$ is a **sheaf** if whenever $x = \vee x_i$ the map

$$Px \longrightarrow \textstyle\prod_j Px_i \rightrightarrows \prod_{i,j} P(x_i \wedge x_j)$$

is an equalizer.

We may collect all the sheaves on L into a category $\mathcal{S}\mathsf{hv}(L)$ with natural transformations as morphisms.

Theorem. $\mathcal{S}\mathsf{hv}(L)$ *is a cccccc.*

In fact, this category is a Grothendieck topos as we shall soon discover.

Here's something interesting:

Theorem ([MM, §III.8])**.** *For any locale L and a sheaf $S \in \mathcal{S}\mathsf{hv}(L)$, the subobject lattice of S is a locale.*

Theorem-Exercise. *The locale L is recovered as the subobject locale of the terminal object.*

Hint. The objects in the image of the Yoneda embedding

$$L \xhookrightarrow{よ} \mathbb{2}^{L^{\mathrm{op}}} \hookrightarrow \mathcal{S}\mathsf{et}^{L^{\mathrm{op}}}$$

are sheaves. I.e., the topology used to define sheaves is "canonical" meaning that it's designed to make the representables into sheaves. Each representable is a subobject of $\mathbb{1}$. So to prove the theorem you show that each subobject of $\mathbb{1}$ is representable and then the Yoneda embedding describes this locale. □

This is special to localic topox.

Theorem ([MM, §III.5])**.** *A topos $\mathcal{E}$ is equivalent to $\mathcal{S}\mathsf{hv}(L)$ for some locale if $\mathcal{E}$ is generated under colimits by subobjects of 1.*

Q. What about morphisms?

Morphisms of locales $f\colon L \to L'$ come with left exact left adjoints. Immediately restricting along the left adjoint $f^{\leftharpoondown}$ defines a map $\mathcal{P}\mathsf{sh}(L) \to \mathcal{P}\mathsf{sh}(L')$. Left exactness will guarantee that it restricts to a morphism $\mathcal{S}\mathsf{hv}(L) \to \mathcal{S}\mathsf{hv}(L')$.

Theorem ([J∞, C.2.3.4])**.** $\mathcal{S}\mathsf{hv}f\colon \mathcal{S}\mathsf{hv}(L) \to \mathcal{S}\mathsf{hv}(L')$ *has a left exact left adjoint!*

Thus we've categorified the notion of morphism of locale to the notion of geometric morphism of localic topos.

Theorem ([MM, §IX.5.2])**.** $\mathcal{L}\mathsf{oc}(L, L') \cong \mathcal{T}\mathsf{opox}(\mathcal{S}\mathsf{hv}(L), \mathcal{S}\mathsf{hv}(L'))$.

References

[J1] P.T. Johnstone The point of pointless topology

[J2] P.T. Johnstone *Stone Spaces*

[J∞] P.T. Johnstone *Sketches of an Elephant*

[MM] Saunders Mac Lane and Ieke Moerdijk *Sheaves in Geometry and Logic: a First Introduction to Topos Theory* Springer, 1992.

[PP] Jorge Picado and Ales Pultr *Frames and Locales: Topology without points*