∞-OPERADS AS ANALYTIC MONADS

NOTES BY EMILY RIEHL

ABSTRACT. The aim is to study the paper "∞-operads as analytic monads" by David Gepner, Rune Haugseng, and Joachim Kock.

CONTENTS

0. Does knowledge of 1-category theory provide morally sufficient grounds upon which to fake knowledge of ∞-category theory? — Emily Riehl

> This talk will proffer an ethical tactic for engaging with ∞-categories as a non-expert. It will start by explaining exactly what an ∞-category is from the point of view of much of the literature that works with them. Along the way, it will also illustrate the similarities and differences between 1-categories and ∞-categories by giving an in-depth discussion of one of the equivalences between ∞-categories that is used without comment in [GHK].

Q. What is an ∞-category?

A 1-category has a collection of objects and then a collection of morphisms between any ordered pair of objects, with an associative and unital binary composition operation defined whenever co/domains align. An equivalence of 1-categories need not induce a bijection on objects, so in some sense "the set of objects" in a category is not really well-defined when working with categorical constructions up to equivalence. An equivalence does define a local bijection on collections of morphisms with fixed co/domains and a global bijection on isomorphism classes of objects.

An *∞-category* has a collection of objects and then a *space* of morphisms between any ordered pair objects, with a weakly associate and weakly unital composition operation defined up to a contractible space of choices whenever co/domains align. The *homotopy category* of an ∞-category is a 1-category obtained by replacing each mapping space with its set of path

Date: Talks in the Johns Hopkins Category Theory Seminar, Spring 2019.

components. An equivalence of ∞-categories induces a local equivalence of mapping spaces and a global bijection on isomorphism classes of objects in the homotopy category. So neither the set of objects in an ∞-category nor the space of morphisms between two objects is well-defined — though the homotopy types of mapping spaces are preserved by equivalences.

This sketch of the idea of an ∞-category as a "category weakly enriched in spaces" can be made precise by various *models*, to which we now turn.

Q. What are the "shortcomings ...afflicting topological categories when viewed as a model for ...∞-categories"?

We've just declared that an ∞-category is a category weakly enriched in spaces. Somewhat surprisingly, and certainly non-obviously, every ∞-category can be modeled by a category strictly enriched in space, i.e., by either a topological category or by a simplicial category. This result can be interpreted as some sort of coherence theorem for ∞-categories.

However, it is not the case that any functor between ∞-categories can be represented by a topologically-enriched functor between the topological categories that represent its domain and codomain: ∞-functors correspond to "homotopy coherent" functors, with the strictly enriched functors being a special case.

And more generally one would like to access the space or even better the ∞-category of ∞-functors between a fixed pair of ∞-categories. This can be constructed for topologically enriched categories[1] but not natural with respect to composition of ∞-functors. This is related to properties of Bergner's model structure for simplicial categories [Ber], which is not cartesian closed and in which relatively few objects are both fibrant and cofibrant.

Q. What is a "better-behaved model for ∞-categories"?

The most popular model of ∞-categories, because it is the simplest to get up and running, is the aforementioned *weak Kan complexes*, now called *quasi-categories*.

defn. A **quasi-category** is a "simplicial set with composition": a simplicial set A in which every inner horn can be filled to a simplex.

$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d, hook] & A \\ \Delta[n] \arrow[ur, dashed] \end{tikzcd} \qquad n \geq 2, 0 < k < n$$

Note that this is weaker than the lifting property which characterizes Kan complexes. In particular, Kan complexes are examples of quasi-categories, as ∞-groupoids are instances of ∞-categories.

The vertices of a quasi-category represent its objects, as an ∞-category, and the edges represents its 1-arrows. By extending along the horn inclusion $\Lambda^1[2] \hookrightarrow \Delta[2]$, any composable pair of arrows admits a composite, with each composition relation witnessed by a 2-simplex; degenerate 2-simplices can be used to witness the identity axioms. The higher horn filling conditions imply that this operation is associative and homotopically unique: the fibers of the right-hand vertical map

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & A^{\Delta[2]} \arrow[d, two heads] \\ \Delta[0] \arrow[r, "{(g,f)}"'] & A^{\Lambda^1[2]} \end{tikzcd}$$

[1] One of the earliest examples of an ∞-category was the "weak Kan complex" of homotopy coherent functors studied by Boardman and Vogt [BV].

are contractible Kan complexes: hence any composable pair of arrows (g, f) has a unique composite, up to homotopy.

Arbitrary maps of simplicial sets $A \to B$ preserve objects, arrows, and composition, and hence are regarded as ∞-functors between ∞-categories. The nerve of a 1-category J always defines a quasi-category; since the nerve embedding is fully faithful, it is increasingly common to use the same notation for the 1-category and the corresponding ∞-category. By examination, the data involved in a simplicial map $J \to A$ assembles into a "homotopy coherent diagram of shape J in A." If A is a quasi-category and J is any simplicial set, then the internal hom $\mathsf{Fun}(J, A) \coloneqq A^J$ is again a quasi-category, defining a natural model for the ∞-category of ∞-functors from J to A.[2]

Q. How should one approach ∞-categories assuming only a background in 1-categories?

In an ideal world, where every mathematician had an unlimited amount of unencumbered free time and could pause any conversation or research project to go read all the relevant literature, we would quote a theorem that we didn't know how to prove. But in the real world, it's desirable to figure out a way to interact with new technology without necessarily understanding how everything works under the hood.

In this vein, many learners, users, or experts in abstract homotopy theory are wondering to what extent their familiarity with 1-category theory can serve as a proxy for understanding of ∞-category theory. In the interest of welcoming as many people as possible into the conversation, I think it's important to avoid having too high a cost of admission into this new direction the field is taking.

I'd advocate approaching ∞-categories with a mix of confidence (that the theorems and constructions that you know and love from 1-category theory likely extend fully faithfully to ∞-categories) and humility (that if you don't happen to know the details of a particular extension, it's likely the case that someone has had or will have to work quite hard to nail them down). In particular, it does not help the field advance if you write a paper asserting that some ∞-categorical fact is analogous to the corresponding 1-categorical fact if you have no idea how one would prove the ∞-categorical version of the theorem you'd like to use.

In talks, the ethical standard is somewhat different, because when trying to tell a coherent story in a constrained amount of time it's often advisable to suppress certain details. Here I see no issue with arguing by analogy with 1-categories — provided that in doing so you do no harm. In this context, harm is caused by intimidating members of the audience into thinking that they're the only ones who don't understand what's going on, for instance by parroting stuff that you don't understand either. If you tend to be under-confident when sketching mathematical proofs then it's likely that your use of language when discussing ∞-categories will naturally be reassuring. But if you tend to be over-confident, you should take care to make sure you don't inadvertently put people off.[3]

Q. How does ∞-categorical methodology differ from 1-categorical methodology?

Part of what confuses readers familiar with 1-categories when reading papers that use ∞-categories is that the arguments that appear seem less rigorous, or at least less explicit. Some of this is because it is considerably more difficult to give "full details" in a new area whose foundations haven't been fully streamlined and sublimated, but some of this is due to a genuine methodological difference in working model-independently with ∞-categories vs working with a theory of 1-categories ultimately grounded in set theory, as we now explain.

---

[2]It is convenient to consider quasi-category-valued diagrams indexed by an arbitrary simplicial set, like it can be convenient to consider category-valued diagrams indexed by a directed graph.

[3]I.e., try not to become one of what I've heard to referred to as "infinity blah blah blah" people.

Some definitions and constructions in 1-category theory are "evil," failing to be invariant under equivalences of categories. To be justifiably considered as an aspect of "∞-category theory" (rather than "quasi-category theory" or "complete Segal space theory") a construction must be invariant under equivalence of ∞-categories because the various "change-of-model functors" only respect equivalence classes of ∞-categories. So, for instance, any particular ∞-category may be introduced by specifying a member of the correct equivalence class, in any model, as we shall now do.

Q. What is $\mathcal{S}$?

By convention $\mathcal{S}$ denotes "the" ∞-category of spaces, well-defined up to equivalence. This mostly naturally arises as a topologically enriched category, or for technical reasons, by considering the full Kan-complex-enriched subcategory of small Kan complexes in the category of simplicial sets. The homotopy coherent nerve of this then defines a large quasi-category which we denote by $\mathcal{S}$. Objects of $\mathcal{S}$ are Kan complexes and morphisms are simplicial functors. Higher simplices represent homotopy coherent diagrams of Kan complexes.

Q. For $f\colon I \to J$ in $\mathcal{S}$ how is

$$\begin{tikzcd} \mathcal{S}_{/I} \arrow[r, bend left, "f_!"] \arrow[r, bend right, "f_*"'] & \mathcal{S}_{/J} \arrow[l, "f^*"'] \end{tikzcd} \qquad \text{equivalent to} \qquad \begin{tikzcd} \mathsf{Fun}(I,\mathcal{S}) \arrow[r, bend left, "\mathsf{lan}_f"] \arrow[r, bend right, "\mathsf{ran}_f"'] & \mathsf{Fun}(J,\mathcal{S}) \arrow[l, "f^*"'] \end{tikzcd} \qquad ?$$

The equivalence $\mathcal{S}_{/J} \simeq \mathsf{Fun}(J,\mathcal{S})$ is very special to the ∞-category $\mathcal{S}$ as we shall explain. It is most easily described when the ∞-category of spaces $\mathcal{S}$ is modeled as a quasi-category, defined as the homotopy coherent nerve of the category of Kan complexes. An element $J \in \mathcal{S}$ is then a small Kan complex.

Since J and $\mathcal{S}$ are both simplicial sets (with J being small and $\mathcal{S}$ being large), the quasi-category $\mathsf{Fun}(J,\mathcal{S}) := \mathcal{S}^J$ may be defined as above: objects are simplicial maps $J \to \mathcal{S}$, morphisms are simplicial natural transformations $J \times \Delta[1] \to \mathcal{S}$, and higher simplices are diagrams $J \times \Delta[n] \to \mathcal{S}$.

The quasi-category $\mathcal{S}_{/J}$ is defined by Joyal's slice construction [Joy], which can be implemented for any vertex in any simplicial set. Objects in $\mathcal{S}_{/J}$ are edges in $\mathcal{S}$ with codomain J. In general, n-simplices in $\mathcal{S}_{/J}$ are $n+1$-simplices in $\mathcal{S}$ with final vertex J. Since $\mathcal{S}$ is a homotopy coherent nerve, this can be unpacked further: objects in $\mathcal{S}_{/J}$ are maps of Kan complexes $X \to J$, while an arrow is comprised of a diagram

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & \arrow[d, phantom, "\Uparrow"] & Y \arrow[dl] \\ & J & \end{tikzcd}$$

given by a trio of simplicial maps forming the boundary of a simplicial natural transformation $X \times \Delta[1] \to J$.

It's not obvious from these explicit descriptions how these quasi-categories are equivalent. The relevant 1-categorical theorem is:

Theorem. *For any 1-category J, there is an equivalence of categories*

$$\mathsf{Fun}(J,\mathsf{Set}) \simeq \mathsf{DLFib}_{/J} \subset \mathsf{Cat}_{/J}$$

between the category of functors $J \to \mathsf{Set}$ and the category of discrete left fibrations over J, a full subcategory of $\mathsf{Cat}_{/J}$.

A **discrete left fibration** is a functor $p\colon E \to J$ with the unique right lifting property

$$\begin{tikzcd} \mathbb{1} \arrow[r] \arrow[d, hook, "\mathrm{dom}"'] & E \arrow[d, "p"] \\ \mathbb{2} \arrow[r] \arrow[ur, dashed, "\exists!"'] & J \end{tikzcd}$$

The corresponding maps of quasi-categories are called **left fibrations** by Joyal [Joy], which are maps of simplicial sets $p\colon E \to J$ characterized by the right lifting property

$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d, hook] & E \arrow[d, "p"] \\ \Delta[n] \arrow[r] \arrow[ur, dashed, "\exists"'] & J \end{tikzcd} \qquad n \geq 1, 0 \leq k < n$$

which in the case $\Lambda^0[1] \hookrightarrow \Delta[1]$ represents the inclusion of the domain of an arrow. The higher left horn fillers make up for the lack of uniqueness of lifts.

By repeated attaching fillers for left horns to X, any map of Kan complexes $X \to J$ can be replaced by a left fibration $\tilde{X} \to J$ related via an equivalence $X \simeq \tilde{X}$ over J. Thus $\mathcal{S}_{/J}$ is equivalent to the full subcategory spanned by the left fibrations over J. Since all of ∞-category is equivalence invariant, there is no need to replace $\mathcal{S}_{/J}$ by this subcategory. Instead, we are free to consider $\mathcal{S}_{/J}$ itself as *the* ∞-category of left fibrations over J.

By a theorem of Joyal [Joy], the fibers of a left fibration $p\colon X \to J$ are Kan complexes. Thus p may be regarded as a J-indexed family of Kan complexes. Moreover, the fact that p is a *left* fibration implies that these fibers vary covariantly functorially in morphisms in the ∞-category J. In this way, a left fibration over J morally corresponds to a ∞-functor $J \to \mathcal{S}$. The ∞-groupoid variant of the straightening-unstraightening theorem of Lurie [L1] then establishes an equivalence

$$\mathcal{S}_{/J} \simeq \mathsf{Fun}(J, \mathcal{S}),$$

which can be understood in a very explicit way. Cisinski [Cis] proves that there is a universal left fibration $u\colon \mathcal{S}_* \to \mathcal{S}$ using techniques similar to those used to establish universal fibrations in presheaf models of homotopy type theory. Here $\mathcal{S}_*$ is the ∞-category of pointed spaces and u is the evident forgetful functor. From right to left, the "unstraightening" of an ∞-functor $F\colon J \to \mathcal{S}$ is given by forming the pullback

$$\begin{tikzcd} \int F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{S}_* \arrow[d, "u"] \\ J \arrow[r, "F"'] & \mathcal{S} \end{tikzcd}$$

of quasi-categories. See [RV] for more details about how this mapping on objects is extended to an ∞-functor.

Finally, note that the construction of the unstraightening of a functor $J \to \mathcal{S}$ is natural in the indexing category, with the unstraightening of a composite diagram $Ff\colon I \to J \to \mathcal{S}$ formed by pulling back $\int F \to J$ along f. Thus, the equivalence $\mathcal{S}_{/J} \simeq \mathsf{Fun}(J, \mathcal{S})$ identifies the pullback and pre-composition functors f^* and $- \circ f$ appearing in the middle of the triple of adjoints. An adjoint to an ∞-functor is well-defined up to natural isomorphism, so it follows that $f_!$ is equivalent to lan_f and f_* is equivalent to ran_f.

1. Introduction — Noah Chrein

The notion of an operad has been around for decades now, going back to May in the 70s. They are used to capture the computational combinatorics of algebraic structures in various situations. Classically, operads are defined using symmetric sequences of sets, these sequences give rise to "analytic endofunctors" which are monads when the symmetric sequence is an operad. For sets, there is an equivalence between the notion of an algebra on an operad and the algebra on its associated monad. Lifting to operads defined as sequences of spaces, this algebraic equivalence is lost in general. Gepner, Haugseng, and Kock notice this problem has to do with higher structures and so devise a definition of an ∞-operad as an analytic monad to recover an analog of the algebraic equivalence in the setting of higher category theory. Several other equivalent models of ∞-operads exist, for example as dendroidal Segal spaces by Cisinski and Moerdijk. GHK's final result proves their analytic model of infinity operads is equivalent to the dendroidal Segal spaces. This introduction will focus on the main constructions and results of the paper that this seminar aims to study in detail during the remainder of the semester.

defn. A **symmetric sequence** in C is a functor $O\colon \mathcal{F}\mathsf{in} \to C$, where $\mathcal{F}\mathsf{in}$ is the ∞-category of finite sets and bijections.

From a symmetric sequence one can define an endofunctor $T\colon C \to C$ by

$$T(X) \coloneqq \coprod_n O(n) \times_{\Sigma_n} X^n,$$

assuming C has these limits and colimits, and when O is an operad, T is a monad. Moreover, **algebras** for the operad, objects $V \in C$ with structure maps $O(n) \times V^n \to V$ correspond to algebras for the monad, which have structure maps $\alpha\colon T(V) \to V$.

When $C = \mathsf{Set}$, we can recover the operad O from from the category of algebras for T, but this isn't true when $C = \mathsf{Spaces}$.

The main result of this paper is that ∞-operads are analytic monads, that is analytic monads are equivalent to the exist models of ∞-operads due to Lurie and due to Cisinski-Moerdijk (as dendroidal Segal spaces).

Now let's give an overview of the sections of the paper.[4]

1.2. **Polynomial functors.** Let $\mathcal{S}$ be the ∞-category of spaces, or of ∞-groupoids. Some key facts:

- $\mathcal{S}$ is an ∞-topos
- $\mathcal{S}$ is locally cartesian closed as an ∞-category, meaning for every $f\colon I \to J$ in $\mathcal{S}$ there exists an adjoint triple

$$\begin{tikzcd} \mathcal{S}_{/I} \arrow[r, bend left, "f_!"] \arrow[r, bend right, "f_*"'] & \mathcal{S}_{/J} \arrow[l, "f^*"'] \end{tikzcd}$$

Let $f\colon I \to J$ be a morphism in $\mathcal{S}$.

[4] The content of the talk has been reordered somewhat to optimize for note-taking.

defn. A **polynomial functor** $P\colon \mathcal{S}_{/I} \to \mathcal{S}_{/J}$ is a functor that arises as $P = t_! p_* s^*$ for some **polynomial**

$$I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} J$$

in $\mathcal{S}$.

There is a geometric realization functor $|-|\colon \mathsf{Cat}_\infty \to \mathcal{S}$ from the ∞-category of ∞-categories to the ∞-category of spaces that freely inverts all of the morphisms in C. An ∞-category C is **weakly contractible** if $|C|$ is weakly contractible.

Theorem. *For a functor $F\colon \mathcal{S}_{/I} \to \mathcal{S}_{/J}$, the following are equivalent:*

(i) F is a polynomial functor
(ii) F is accessible and preserves weakly contractible limits
(iii) F is a local right adjoint.

defn. A functor $F\colon C \to D$ is a **local right adjoint** if for all $x \in C$ the functor $F_{/x}\colon C_{/x} \to D_{/x}$ is a right adjoint for all $x \in C$.

Remark. By the adjoint functor theorem, a functor between locally presentable ∞-categories is a left adjoint if and only if it preserves colimits and is a right adjoint if and only if it is accessible and preserves limits.

For any $I, J \in \mathcal{S}$, there is an ∞-category $\mathcal{P}\mathsf{olyFun}(I, J)$ whose objects are polynomial functors from I to J and whose morphisms are **cartesian natural transformations**, natural transformations whose naturality squares are pullback squares.

Lemma. *If $\eta\colon F \to P$ is a cartesian natural transformation whose codomain P is a polynomial functor, then F is a polynomial functor.*

The definition of the ∞-category $\mathcal{P}\mathsf{olyFun}$ of polynomial functors with varying endpoints is somewhat complicated.

The category $\mathcal{P}\mathsf{oly}$ is defined as a subcategory of diagrams of shape $\bullet \leftarrow \bullet \to \bullet \to \bullet$ in $\mathcal{S}$ containing all objects and only those morphisms whose "middle square" as below is a pullback

$$\begin{tikzcd}
\bullet \arrow[d] & \bullet \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\
\bullet & \bullet \arrow[l] \arrow[r] & \bullet \arrow[r] & \bullet
\end{tikzcd}$$

Theorem. *The ∞-categories $\mathcal{P}\mathsf{oly}$ and $\mathcal{P}\mathsf{olyFun}$ are equivalent.*

1.3. **Analytic functors.**

defn. An **analytic functor** $F\colon \mathcal{S}_{/I} \to \mathcal{S}_{/J}$ preserves weakly contractible limits and sifted colimits.

This is a strengthening of the second characterization of polynomial functors in the theorem above, since filtered colimits are sifted.[5] Consequently:

Corollary. *Analytic functors are polynomial functors.*

Theorem. *Analytic functors are those polynomial functor represented by polynomials*

$$I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} J$$

in which p has finite discrete fibers.

[5] *Filtered* colimits commute with finite limits; *sifted* colimits commute with finite products.

Proposition. *Let $\mathcal{F}$ be a "bounded local class of morphisms" in $\mathcal{S}$ with classifying family $U'_{\mathcal{F}} \to U_{\mathcal{F}}$, and let $F\colon \mathcal{S} \to \mathcal{S}$ be the polynomial functor corresponding to this map. Then the forgetful functor*

$$\mathcal{P}\mathsf{olyFun}_{/F} \longrightarrow \mathcal{P}\mathsf{olyFun}$$

is fully faithful and its image is the full subcategory $\mathcal{P}\mathsf{olyFun}_{/\mathcal{F}}$ of polynomial functors represented by polynomials whose middle map is in the class $\mathcal{F}$.

An example is the family $\mathcal{F}$ of maps whose fibers are finite sets. Then by the previous theorem, $\mathcal{A}\mathsf{nFun} \simeq \mathcal{P}\mathsf{olyFun}_{/\mathcal{F}} \simeq \mathcal{P}\mathsf{olyFun}_{/E}$ where E is the polynomial functor with polynomial

$$* \longleftarrow \iota\mathcal{F}\mathsf{in}_* \longrightarrow \iota\mathcal{F}\mathsf{in} \longrightarrow *$$

Here $\mathcal{F}\mathsf{in}$ is the category of finite sets and all maps while $\iota\mathcal{F}\mathsf{in}$ is its maximal subgroupoid, the category of finite sets and isomorphisms. This proves:

Theorem. *If F is analytic then its polynomial is a pullback*

$$\begin{tikzcd} I \arrow[d] & E \arrow[l, "s"'] \arrow[r, "p"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[r, "t"] \arrow[d] & J \arrow[d] \\ * & \iota\mathcal{F}\mathsf{in}_* \arrow[l] \arrow[r] & \iota\mathcal{F}\mathsf{in} \arrow[r] & * \end{tikzcd}$$

Consequently

$$F(X) = \coprod_n (B_n \times_{\Sigma_n} X^n)$$

where the B_n are the fibers of $B \to \iota\mathcal{F}\mathsf{in}$.

1.4. **Initial Algebras and Free Monads.** Let P be an endofunctor of some ∞-category $\mathcal{C}$. A P-**algebra** is a pair (A, a) with $a\colon PA \to A$. Similarly a P-**coalgebra** is a pair (C, c) where $c\colon C \to PC$.

For any P-coalgebra one can define the colimit

$$\Omega C = \mathrm{colim}(\ C \xrightarrow{c} PC \xrightarrow{Pc} P^2C \longrightarrow \cdots\ ).$$

It turns out that ΩC is a P-algebra so this defines a functor

$$\Omega\colon \mathsf{Coalg}_P(\mathcal{C}) \to \mathsf{Alg}_P(\mathcal{C}).$$

Dually one defines

$$B\colon \mathsf{Alg}_P(\mathcal{C}) \to \mathsf{Coalg}_P(\mathcal{C}),$$

and these are adjoint $\Omega \dashv B$, defining the **bar-cobar adjunction**.

The category of P-algebras has a free-forgetful adjunction inducing a monad $\bar{P}$ on $\mathcal{C}$, and this defines a functor

$$\mathrm{Fr}\colon \mathsf{End}(\mathcal{C}) \to \mathsf{Mon}(\mathcal{C}),$$

by $\mathrm{Fr}(P) = \bar{P}$.

When this construction applied to analytic endofunctors it gives an analytic monad, and this defines a left adjoint to the inclusion

$$\begin{tikzcd} \mathsf{AnEnd}(I) \arrow[r, bend left, "\mathrm{Fr}"] \arrow[r, phantom, "\perp"] & \mathsf{AnMnd}(I) \arrow[l, bend left, "U"] \end{tikzcd}$$

Moreover this adjunction is monadic.

Consequently, the category of analytic monads is equivalent to the category of algebras for the monad $U\mathrm{Fr}$ on the category of endofunctors. What this means in practice is that from an algebra on an analytic endofunctor we can recover our analytic monad.

1.5. **Analytic monads and ∞-operads.** In the final section they show that analytic monads are Segal presheaves on the category of trees. This is the Cisinski-Moerdijk notion of ∞-operad.

2. Polynomial Functors — Martina Rovelli

The main character is $\mathcal{S}$, the ∞-category of spaces. For 90% of what follows, you can also take $\mathcal{S}$ to be the category of sets.

For any $I \in \mathcal{S}$ you can construct the slice ∞-category $\mathcal{S}_{/I}$ whose objects are $f\colon X \to I$. Note that $\mathcal{S}_{/I}$ has a terminal element, namely the identity at I.

Each $f\colon I \to J \in \mathcal{S}$ gives rise to an adjoint triple

$$\begin{tikzcd} \mathcal{S}_{/I} \arrow[r, bend left, "f_!"] \arrow[r, bend right, "f_*"'] & \mathcal{S}_{/J} \arrow[l, "f^*" description] \end{tikzcd}$$

The left adjoint $f_!$ is composition with f aka the dependent sum Σ_f. The middle functor f^* is pullback along f. The right adjiont f_* is called the dependent product and also denoted by Π_f. As a right adjoint $f_*(\mathrm{id}_I) = \mathrm{id}_J$.

defn. A **polynomial** is a diagram

$$I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} J$$

To each polynomial you can associate a functor

$$P\colon \mathcal{S}_{/I} \xrightarrow{s^*} \mathcal{S}_{/E} \xrightarrow{p_*} \mathcal{S}_{/B} \xrightarrow{t_!} \mathcal{S}_{/J}$$

Q. When does $F\colon \mathcal{S}_{/I} \to \mathcal{S}_{/J}$ arise in this manner?

The following theorem characterizes **polynomial functors**.

Theorem. *For $F\colon \mathcal{S}_{/I} \to \mathcal{S}_{/J}$ TFAE*

- *(i) $F = t_! p_* s^*$ for some polynomial.*
- *(ii) F is accessible and preserves weakly contractible limits.*
- *(iii) F is a local right adjoint: meaning functors induced by F on slices over an object in $\mathcal{S}_{/I}$ are right adjoints.*

A functor is **accessible** if it preserves κ-filtered colimits for some regular cardinal κ. **Weakly contractible limits** are limits indexed by categories whose geometric realization is weakly equivalent to a point.

A related result will help prove the main theorem.

Theorem. *For $F\colon \mathcal{S}_{/I} \to \mathcal{S}_{/B}$ TFAE*

- *(i) $F = p_* s^*$ for some polynomial.*
- *(ii) F is accessible and preserves all limits.*
- *(iii) F is a right adjoint.*

Proof. The equvialence $(ii) \Leftrightarrow (iii)$ is the adjoint functor theorem. Clearly $(i) \Rightarrow (iii)$.

For $(iii) \Rightarrow (i)$ the assignment $(I \xleftarrow{s} E \xrightarrow{p} B) \mapsto (\mathcal{S}_{/I} \xrightarrow{s^*} \mathcal{S}_{/E} \xleftarrow{p_*} \mathcal{S}_{/B})$ is part of an equivalence and can be described as the following:

$$\mathcal{S}_{/I\times B} \simeq \mathsf{Fun}(I \times B, \mathcal{S}) \simeq \mathsf{Fun}(B, \mathsf{Fun}(I, \mathcal{S})) \simeq \mathsf{Fun}^L(\mathsf{Fun}(B, \mathcal{S}), \mathsf{Fun}(I, \mathcal{S})),$$

where this last equivalence expresses the universal property of the free colimit completion $よ\colon B \to \mathsf{Fun}(B, \mathcal{S})$, ignoring ops (since B is a space), and then since $\mathsf{Fun}(I, \mathcal{S}) \simeq \mathcal{S}_{/I}$ we have

$$\simeq \mathsf{Fun}^L(\mathcal{S}_{/B}, \mathcal{S}_{/I}) \simeq \mathsf{Fun}^R(\mathcal{S}_{/I}, \mathcal{S}_{/B}).$$ □

Proof of the main theorem. (i)⇒(ii) is a direct verification. (ii)⇒(iii) again involves the adjoint functor theorem.

The interesting implication is (iii)⇒(i). Suppose $F\colon \mathcal{S}_{/I} \to \mathcal{S}_{/J}$ is a local right adjoint. Now consider the sliced functor

$$\mathcal{S}_{/I} \simeq (\mathcal{S}_{/I})_{/\mathrm{id}_I} \xrightarrow{F_{/\mathrm{id}_I}} (\mathcal{S}_{/J})_{/F(\mathrm{id}_I)}$$

If $F(\mathrm{id}_I) = Y \to J$ then we have an equivalence between the double slice and the single slice:

$$\begin{tikzcd} \mathcal{S}_{/I} \simeq (\mathcal{S}_{/I})_{/\mathrm{id}_I} \arrow[r, "F_{/\mathrm{id}_I}"] & (\mathcal{S}_{/J})_{/F(\mathrm{id}_I)} \arrow[l, bend left, dashed, "\perp"', "R"] \simeq \mathcal{S}_{/Y} \arrow[r, "F(\mathrm{id}_I)_!"] & \mathcal{S}_{/J} \end{tikzcd}$$

By the previous result R has the form $p_* s^*$. So now this composite, which is F again, is $t_! p_* s^*$. □

Composition of polynomial functors. By the second condition of the theorem, the composite of polynomial functors is a polynomial functor. But what is the composite polynomial? There are two main tools we'll use to answer this question.

defn (Beck-Chevalley transformations)**.** Given a square

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "g"'] & B \arrow[d, "f"] \\ C \arrow[r, "v"'] & D \end{tikzcd}$$

we can construct

$$\begin{tikzcd} \mathcal{S}_{/A} & \mathcal{S}_{/B} \arrow[l, "u^*"'] \\ \mathcal{S}_{/C} \arrow[u, "g^*"] & \mathcal{S}_{/D} \arrow[l, "v^*"] \arrow[u, "f^*"'] \end{tikzcd}$$

Pasting with the counit of $u_! \dashv u^*$ and $v_! \dashv v^*$ we get

$$\begin{tikzcd} \mathcal{S}_{/A} \arrow[r, "u_!"] \arrow[dr, phantom, "\Searrow"] & \mathcal{S}_{/B} \\ \mathcal{S}_{/C} \arrow[u, "g^*"] \arrow[r, "v_!"'] & \mathcal{S}_{/D} \arrow[u, "f^*"'] \end{tikzcd}$$

and dually

$$\begin{tikzcd}
\mathcal{S}_{/A} \arrow[d, "g_*"'] \arrow[dr, phantom, "\Nwarrow"] & \mathcal{S}_{/B} \arrow[l, "u^*"'] \arrow[d, "f_*"] \\
\mathcal{S}_{/C} & \mathcal{S}_{/D} \arrow[l, "v^*"]
\end{tikzcd}$$

Lemma. *The original square is cartesian if and only if these induced transformations are invertible.*

The upshot is that you can always push a lower shrieck to the left of an upper star by taking the pullback of the cospan to form the cartesian square. Dually, you can always move a lower star to the right of an upper star.

What's missing is a way to swap $(-)_*$ and $(-)_!$. The answer is not as good but almost: you get an extra term which is an upper star, but this is okay because you know how to handle them.

Lemma (distributivity)**.** *Given $E \xrightarrow{g} X \xrightarrow{f} Y$ you can form a diagram*

$$\begin{tikzcd}
E \arrow[dr] & E' \arrow[l, "\epsilon"'] \arrow[r, "q"] & E'' \arrow[d, "p"] \\
& X \arrow[r] & Y
\end{tikzcd}$$

by forming $f_*(g)$ and then $f^*f_*(g)$ so that

$$\begin{tikzcd}
\mathcal{S}_{/E} \arrow[r, "\epsilon^*"] \arrow[dr, "g_*"'] & \mathcal{S}_{/E'} \arrow[r, "q_!"] & \mathcal{S}_{/E''} \arrow[d, "p_*"] \\
& \mathcal{S}_{/X} \arrow[r, "f_!"'] & \mathcal{S}_{/Y}
\end{tikzcd}$$

commutes.

Now we can compute the composite. Given

$$\begin{tikzcd}
& E \arrow[r] \arrow[dl] & B \arrow[dr] & & F \arrow[r] \arrow[dl] & C \arrow[dr] & \\
I & & & J & & & K
\end{tikzcd}$$

build

$$\begin{tikzcd}
& & & G \arrow[r] \arrow[dl] \arrow[dr, phantom, very near start, "\lrcorner"] & X \arrow[r] \arrow[dl] & D \arrow[dd] & \\
& & Y \arrow[r] \arrow[dl] \arrow[dr, phantom, very near start, "\lrcorner"] & B \times_J F \arrow[dl] \arrow[dr] & & & \\
& E \arrow[r] \arrow[dl] & B \arrow[dr] & & F \arrow[r] \arrow[dl] & C \arrow[dr] & \\
I & & & J & & & K
\end{tikzcd}$$

Then

$$\begin{tikzcd}
 & & & \mathcal{S}_{/G} \arrow[r] \arrow[rr, dashed, bend left] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{S}_{/X} \arrow[r] \arrow[dl] & \mathcal{S}_{/D} \arrow[dd] \arrow[dddr, dashed, bend left] & \\
 & & \mathcal{S}_{/Y} \arrow[ur] \arrow[r] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{S}_{/B\times_J F} \arrow[dl] \arrow[dr] & & & \\
 & \mathcal{S}_{/E} \arrow[r] \arrow[ur] & \mathcal{S}_{/B} \arrow[dr] & & \mathcal{S}_{/F} \arrow[r] & \mathcal{S}_{/C} \arrow[dr] & \\
\mathcal{S}_{/I} \arrow[ur] \arrow[uuurrr, dashed, bend left] & & & \mathcal{S}_{/J} \arrow[ur] & & & \mathcal{S}_{/K}
\end{tikzcd}$$

The dashed composites are the three components of the composite polynomial functor.

Morphisms of polynomials. A natural transformation

$$\begin{tikzcd} \mathcal{S}_{/I} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{S}_{/J} \end{tikzcd}$$

is **cartesian** if all naturality squares are pullback squares. Equivalently, for all $f\colon X \to I$ in $\mathcal{S}_{/I}$ the square

$$\begin{tikzcd} Ff \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Gf \arrow[d] \\ F\mathrm{id}_I \arrow[r] & G\mathrm{id}_I \end{tikzcd}$$

is a pullback.

Why cartesian morphisms are cool:

- α is an equivalence if and only if α_{id_I} is an equivalence.
- Given $\alpha\colon F \Rightarrow G$ and $\beta\colon K \Rightarrow G$ such that $\alpha_{\mathrm{id}_I} \simeq \beta_{\mathrm{id}_I}$ then $\alpha \simeq \beta$ which implies in particular that $F \simeq K$.
- Cartesian natural transformations are the cartesian edges for the cartesian fibration

$$\mathrm{cod}\colon \mathsf{Fun}(\mathcal{S}_{/I}, \mathcal{S}_{/J}) \to \mathcal{S}_{/J}$$

defn. The **∞-category $\mathcal{P}\mathsf{olyFun}(I, J)$ of polynomial functors from I to J** is the sub ∞-category of $\mathsf{Fun}(\mathcal{S}_{/I}, \mathcal{S}_{/J})$ whose objects are the polynomial functors $\mathcal{S}_{/I} \to \mathcal{S}_{/J}$ and whose morphisms are the cartesian morphisms (and with all higher cells between them).

Our next aim is to show that the ∞-category $\mathcal{P}\mathsf{olyFun}(I, J)$ is equivalent to an ∞-category $\mathcal{P}\mathsf{oly}(I, J)$ that we'll now introduce.

defn. The **∞-category** of polynomials from I to J is approximately

$$\mathcal{S}_{/I} \times_{\mathcal{S}} \mathsf{Fun}(\mathbb{2}, \mathcal{S}) \times_{\mathcal{S}} \mathcal{S}_{/J}$$

where the pullbacks says that the source of the middle arrow is the source of the arrow whose codomain is J and that the target of middle arrow is the source of the arrow whose codomain is I.

Unpacking this, we see that the morphisms have the form

$$\begin{tikzcd} I & E \arrow[l] \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & J \\ I \arrow[u, equal] & E' \arrow[l] \arrow[r] & B' \arrow[r] & J \arrow[u, equal] \end{tikzcd}$$

except this isn't quite right. In the definition above, you want the middle square to be a pullback

$$\begin{tikzcd} I & E \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[r] \arrow[d] & J \\ I \arrow[u, equal] & E' \arrow[l] \arrow[r] & B' \arrow[r] & J \arrow[u, equal] \end{tikzcd}$$

which you get by taking $\mathsf{Fun}^{\mathrm{cart}}(\mathbb{2}, \mathcal{S}) \hookrightarrow \mathsf{Fun}(\mathbb{2}, \mathcal{S})$ is the subcategory whose morphisms are pullback squares. In fact define

$$\mathcal{P}\mathsf{oly}(I, J) \coloneqq \mathcal{S}_{/I} \times_{\mathcal{S}} \mathsf{Fun}^{\mathrm{cart}}(\mathbb{2}, \mathcal{S}) \times_{\mathcal{S}} \mathcal{S}_{/J}.$$

We now how to take a polynomial to a polynomial functor but to define $\phi_{I,J}\colon \mathcal{P}\mathsf{oly}(I, J) \to \mathcal{P}\mathsf{olyFun}(I, J)$ we also need to take morphisms of polynomials to cartesian transformations. Given a square as above you get a pasting "composite" of two Beck-Chevalley transformations

$$\begin{tikzcd} \mathcal{S}_{/I} \arrow[r] & \mathcal{S}_{/E} \arrow[r] \arrow[dr, phantom, "\Nwarrow"] & \mathcal{S}_{/B} \arrow[r] \arrow[dr, phantom, "\Searrow"] & \mathcal{S}_{/J} \\ \mathcal{S}_{/I} \arrow[u, equal] \arrow[r] & \mathcal{S}_{/E'} \arrow[u] \arrow[r] & \mathcal{S}_{/B'} \arrow[u] \arrow[r] & \mathcal{S}_{/J} \arrow[u, equal] \end{tikzcd}$$

which looks like it shouldn't compose but since the middle square was a pullback the middle map is invertible.

Theorem. *The map* $\phi_{I,J}\colon \mathcal{P}\mathsf{oly}(I, J) \to \mathcal{P}\mathsf{olyFun}(I, J)$ *is an equivalence of* ∞*-categories.*

Why should this be true? We've seen that the objects are the same. Given a cartesian transformation

$$\begin{tikzcd} & \mathcal{S}_{/E} \arrow[r] & \mathcal{S}_{/B} \arrow[dr] & \\ \mathcal{S}_{/I} \arrow[ur] \arrow[dr] & & \Downarrow \alpha & \mathcal{S}_{/J} \\ & \mathcal{S}_{/E'} \arrow[r] & \mathcal{S}_{/B'} \arrow[ur] & \end{tikzcd}$$

Observe that the component α_{id_I} gives a map of the form $B \to B'$ over J. Define P to be the pullback and define a map $P \to E' \to I$ as the composite.

This gives a diagram of the correct type to define a morphism A in $\mathcal{P}\mathsf{oly}(I, J)$ but its source isn't quite right because it involves the object P rather than E. Observe however that $\phi_{I,J}(A)$ and α are cartesian maps between polynomial functors with the same target and the same component at id_I so by the fact above $\phi(A) \simeq \alpha$.

This heuristic argument effectively shows that the ∞-categories $\mathcal{P}\mathsf{oly}(I, J)$ and $\mathcal{P}\mathsf{olyFun}(I, J)$ have the same homotopy category but isn't enough to show that the ∞-categories are equivalent. We'll now give the ingredients of the full proof.

Proof. There are maps

$$\mathcal{P}\mathsf{oly}(I, J) \longrightarrow \mathcal{S}_{/J} \longleftarrow \mathcal{P}\mathsf{olyFun}(\mathcal{S}_{/I}, \mathcal{S}_{/J})$$

defined in the first case by taking the component $B \to J$ of a polynomial $I \leftarrow E \to B \to J$ and defined in the second case by evaluating $F\colon \mathcal{S}_{/I} \to \mathcal{S}_{/J}$ at id_I. Both functors turn out to be right fibrations.

The map $\phi_{I,J}$ commutes with the maps to $\mathcal{S}_{/J}$ so it is enough to show that it is an equivalence on fibers over each object $Y \to J$. For $\mathcal{P}\mathsf{oly}(I,J) \to \mathcal{S}_{/J}$ the fiber looks like $\mathcal{S}_{/I\times Y}$ on objects but not on morphisms: we need to impose the condition that the square on the right is a pullback. But here we're pulling back an identity so that's the same as restricting to the groupoid core $(\mathcal{S}_{/I\times Y})^{\simeq}$. On the other side, by our first theorem characterizing truncated polynomial functors the fiber $\mathsf{Fun}^{R,\mathrm{cart}}(\mathcal{S}_{/I},\mathcal{S}_{/Y})$ is given by functors that are right adjoints, together with cartesian transformations between them. This map is a restriction of the equivalence $\mathcal{S}_{/I\times Y} \simeq \mathsf{Fun}^R(\mathcal{S}_{/I},\mathcal{S}_{/Y})$. This is a bit surprising because it suggests that $\mathsf{Fun}^{R,\mathrm{cart}}(\mathcal{S}_{/I},\mathcal{S}_{/Y})$ is an ∞-groupoid but you can see this by appealing to one of the cartesian transformation facts above: a cartesian transformation is invertible iff its component at the terminal object is invertible and since the domains and codomains are right adjoints they preserve this terminal object, so that component is indeed invertible. □

Morphisms of polynomial functors with varying domains and codomains. Recall

$$\mathcal{P}\mathsf{oly}(I,J) := \mathcal{S}_{/I} \times_{\mathcal{S}} \mathsf{Fun}^{\mathrm{cart}}(\mathbb{2},\mathcal{S}) \times_{\mathcal{S}} \mathcal{S}_{/J}.$$

So to define a corresponding ∞-category with varying endpoints define

$$\mathcal{P}\mathsf{oly} := \mathsf{Fun}(\mathbb{2},\mathcal{S}) \times_{\mathcal{S}} \mathsf{Fun}^{\mathrm{cart}}(\mathbb{2},\mathcal{S}) \times_{\mathcal{S}} \mathsf{Fun}(\mathbb{2},\mathcal{S}).$$

Objects are polynomials as before, while morphisms now look like

$$\begin{tikzcd} I \arrow[d, "a"'] & E \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[r] \arrow[d] & J \arrow[d, "b"] \\ I' & E' \arrow[l] \arrow[r] & B' \arrow[r] & J' \end{tikzcd}$$

Similarly, we can define an ∞-category $\mathcal{P}\mathsf{olyFun}$ whose objects are polynomial functors $\mathcal{S}_{/I} \to \mathcal{S}_{/J}$ for varying I and J and morphisms look like

$$\begin{tikzcd} \mathcal{S}_{/I} \arrow[r, "F"] \arrow[dr, phantom, "\Searrow"] & \mathcal{S}_{/J} \\ \mathcal{S}_{/I'} \arrow[u, "a^*"] \arrow[r, "F'"'] & \mathcal{S}_{/J'} \arrow[u, "b^*"'] \end{tikzcd}$$

Here we're suppressing many details because we haven't explained what the higher cells are.

You can define $\phi\colon \mathcal{P}\mathsf{oly} \to \mathcal{P}\mathsf{olyFun}$ as before, sending the morphism of polynomials above to

$$\begin{tikzcd} \mathcal{S}_{/I} \arrow[r] & \mathcal{S}_{/E} \arrow[r] & \mathcal{S}_{/B} \arrow[r] & \mathcal{S}_{/J} \\ \mathcal{S}_{/I'} \arrow[r] \arrow[u] & \mathcal{S}_{/E'} \arrow[r] \arrow[u] \arrow[ur, phantom, "\Nwarrow"] & \mathcal{S}_{/B'} \arrow[r] \arrow[u] \arrow[ur, phantom, "\Searrow"] & \mathcal{S}_{/J'} \arrow[u] \end{tikzcd}$$

Theorem. *The map* $\phi\colon \mathcal{P}\mathsf{oly} \to \mathcal{P}\mathsf{olyFun}$ *is an equivalence.*

Proof. The endpoint evaluation maps

$$\mathcal{P}\mathsf{oly} \longrightarrow \mathcal{S} \times \mathcal{S} \longleftarrow \mathcal{P}\mathsf{olyFun}$$

are cartesian fibrations and ϕ is a cartesian functor between them. Fiberwise it precisely induces the map $\phi_{I,J}$ which we've shown is an equivalence. □

This covers the first four of six sections in this part. We'll just state the results from the remaining two sections.

Colimits of polynomials. The ∞-categories $\mathcal{P}\mathsf{oly} \simeq \mathcal{P}\mathsf{olyFun}$ and $\mathcal{P}\mathsf{oly}(I,J) \simeq \mathcal{P}\mathsf{olyFun}(I,J)$ are cocomplete and colimits are computed in "easier" ∞-categories:

Theorem. *The following functors create colimits:*

(i) $\mathcal{P}\mathsf{oly} \to \mathsf{Fun}(\bullet \leftarrow \bullet \to \bullet \to \bullet, \mathcal{S})$, *forgetting that the middle square is cartesian.*
(ii) $\mathcal{P}\mathsf{oly}(I,J) \to \mathsf{Fun}(\mathbb{2}, \mathcal{S})$ *that carries a polynomial to the middle map $E \to B$.*
(iii) $\mathcal{P}\mathsf{olyFun}(I,J) \to \mathsf{Fun}(\mathcal{S}_{/I}, \mathcal{S}_{/J})$ *forgetting that your functors are polynomial and your transformations are cartesian.*

Slices of $\mathcal{P}\mathsf{olyFun}$. The category $\mathcal{P}\mathsf{olyFun}$ is not well-behaved because it is not accessible (if it were, it would be presentable by the just-established cocompleteness). However, when you take slices over a fixed polynomial functor, then $\mathcal{P}\mathsf{olyFun}_{/P}$ is an ∞-topos.

Key facts:

- $\mathcal{S}$ is an ∞-topos.
- Diagrams, such as $\mathsf{Fun}(\mathbb{2}, \mathcal{S})$, valued in an ∞-topos is an ∞-topos.
- Slices of an ∞-topos is an ∞-topos.
- The pullback of an ∞-topos along left exact left adjoints is an ∞-topos.

Proof. Consider the slice $\mathcal{P}\mathsf{oly}_{/\mathbb{D}}$ over a polynomial $\mathbb{D} = I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} J$. Then

$$\mathcal{P}\mathsf{oly}_{/\mathbb{D}} \simeq \mathsf{Fun}(\mathbb{2}, \mathcal{S})_{/s} \times_{\mathcal{S}_{/E}} \mathsf{Fun}^{\mathrm{cart}}(\mathbb{2}, \mathcal{S})_{/p} \times_{\mathcal{S}_{/B}} \mathsf{Fun}(\mathbb{2}, \mathcal{S})_{/t}$$

Since morphisms in $\mathsf{Fun}^{\mathrm{cart}}(\mathbb{2}, \mathcal{S})$ are pullback squares $\mathsf{Fun}^{\mathrm{cart}}(\mathbb{2}, \mathcal{S})_{/p} \simeq \mathcal{S}_{/B}$. So

$$\mathcal{P}\mathsf{oly}_{/\mathbb{D}} \simeq \mathsf{Fun}(\mathbb{2}, \mathcal{S})_{/s} \times_{\mathcal{S}_{/E}} \mathcal{S}_{/B} \times_{\mathcal{S}_{/B}} \mathsf{Fun}(\mathbb{2}, \mathcal{S})_{/t}$$

and since these objects are ∞-topoi and the functors are left exact left adjoints, $\mathcal{P}\mathsf{oly}_{/\mathbb{D}}$ is an ∞-topos.

Similarly

$$\mathcal{P}\mathsf{oly}(I,J)_{/\mathbb{D}} \simeq \cdots \simeq \mathcal{S}_{/B}$$

is an ∞-topos. □

Finally, you can define polynomial endofunctors

$$\begin{tikzcd} \mathcal{P}\mathsf{olyEnd} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{P}\mathsf{oly} \arrow[d] \\ \mathcal{S} \arrow[r, "\Delta"'] & \mathcal{S} \times \mathcal{S} \end{tikzcd}$$

and again $\mathcal{P}\mathsf{olyEnd}_{/\mathbb{D}}$ is an ∞-topos.

Remark. Note that $\mathcal{P}\mathsf{oly} \simeq \mathcal{P}\mathsf{olyFun}$ doesn't have a terminal object because if it did the slice over it would be an ∞-topos and so $\mathcal{P}\mathsf{oly}$ would be too.

3. Analytic Functors — David Myers

Generating functorology. tslil told us that sets are numbers: $n \in \mathbb{N}$ corresponds to $\{1, \dots, n\}$; $+$ is disjoint union; $\times$ is cartesian product; and exponentiation corresponds to the set of functions. Note this seems like we're moving from concrete to abstract but really the historical move was in the other direction: from a set of things to the abstract concept of number. What follows will be less historical.

Q. How do we divide sets?

One way to think about division has to do with partitioning sets (into equal size subsets). To that end, suppose a group G acts on a set X freely. Then the set X/G of orbits satisfies $|X/G| = |X|/|G|$. Eg 3 coins = 6 coin faces/2 symmetries.

Now suppose one coin has just one face: 5 coin faces/2 symmetries = 2.5 coins. (Of course this action is no longer free but in math we often like to preserve the formula by modifying the meaning of the terms.) We can think of this 2.5 coins as 1 + 1 + 1/2 where in each case we are adding 1 over the size of the stabilizer of that element.

We can think of this count as having something to do with a groupoid: the **action groupoid** $X//G$ of the action whose objects are X and which has an arrow $g\colon x \to y$ iff $g \cdot x = y$. Note $\pi_0(X//G) = X/G$.

defn. The **cardinality of a finite groupoid** G is

$$\#G = \sum_{x \in \pi_0 G} \frac{1}{\#\mathrm{Aut}(x)}.$$

Remark. Apparently the Euler characteristic of an ∞-groupoid can be defined similarly but now the formula above should be interpreted coinductively: $\mathrm{Aut}(x)$ is the automorphism ∞-groupoid.

Lemma. $\#(X//G) = \#X/\#G$.

For instance G acts on a singleton, so $\#(*//G) = 1/\#G$.

The nerve of $*//G$ is often called BG, which here we take as a simplicial set:

$$* \leftrightarrows G \leftleftarrows G \times G \qquad \cdots$$

We can calculate its Euler characteristic as the alternating sum over the non-degenerate simplices in each dimension:

$$\chi(BG) = 1 - (\#G - 1) + (\#G - 1)^2 - \cdots = \frac{1}{1 + (\#G - 1)} = \frac{1}{\#G}.$$

So the groupoid cardinality is related to the Euler characteristic.

Let $\mathcal{F}\mathrm{in}$ be the groupoid of finite sets. Then $\pi_0 \mathcal{F}\mathrm{in} = \mathbb{N}$ and $\#\mathrm{Aut}\{1, \ldots, n\} = n!$. And the groupoid cardinality is

$$\#\mathcal{F}\mathrm{in} = \sum_{n \in \mathbb{N}} \frac{1}{n!} = e.$$

Aside. So why are so many probabilities $1/e$? Well the average size of a finite set is e so the probability of picking out a thing from that finite set is $1/e$.

defn. A **type of stuff you can put on a finite set** is a functor $\mathcal{F}\mathrm{in} \to \mathcal{S}$ which sends X to the homotopy type of F-stuff on X.

Eg $F(X) = X$ or $F(X)$ = bracketings of elements of X or $F(X) = \mathrm{Aut}(X)$ or $F(X)$ = groupoid simple rings with underlying set X or $F(X)$ = binary trees whose edge set is labelled by X (same as bracketings).

defn. Define a formal power series $\#F(x) = \sum_{n \in \mathbb{N}} \#F(x) \cdot \frac{x^n}{n!}$ using the ∞-groupoid cardinality.

For $F(X) = \mathrm{Aut}(X)$. This gives $\frac{1}{1-x}$. For the simple rings and a different definition of groupoid cardinality this gives the Riemann-Zeta function. For the bracketings you get $\sum c_n x^n$ where c_n are the Catalan numbers.

Q. $\#F(\#X) = \#F(X)$?

Take the left Kan extension $\mathsf{lan}F\colon \mathcal{S} \to \mathcal{S}$ of F along the inclusion $\mathcal{F}\mathsf{in} \hookrightarrow \mathcal{S}$ to interpret the above. Then

$$\mathsf{lan}F(X) = \sum_{n\in\mathbb{N}} F(n) \times X^n /\!/ \Sigma_n$$

and you can read off from the formula that the cardinalities come out right.

What do we call functions that are determined by their power series? Answer: analytic. So functors that are determined by the power series of their stuff types are **analytic functors**.

The functor $I\colon \mathcal{F}\mathsf{in} \hookrightarrow \mathcal{S}$ mapping X to X corresponds to a fibration $\mathcal{F}\mathsf{in}_* \to \mathcal{F}\mathsf{in}$ by

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{F}\mathsf{in}_* \arrow[d, "u"] \\ * \arrow[r, "X"] & \mathcal{F}\mathsf{in} \end{tikzcd}$$

where $\mathcal{F}\mathsf{in}_*$ the groupoid of finite point sets. This is the universal fibration with finite fibers:

$$\begin{tikzcd} E \arrow[r, "\exists 1"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{F}\mathsf{in}_* \arrow[d, "u"] \\ B \arrow[r, "\exists !"] & \mathcal{F}\mathsf{in} \end{tikzcd}$$

So what if the middle map $p\colon E \to B$ of a polynomial has finite fibers. Then you get a diagram

$$\begin{tikzcd} I \arrow[d] & E \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[r] \arrow[d] & J \arrow[d] \\ * & \mathcal{F}\mathsf{in}_* \arrow[l] \arrow[r, "u"] & \mathcal{F}\mathsf{in} \arrow[r] & * \end{tikzcd}$$

So this tells us that the slice of polynomials over this canonical polynomial $u\colon \mathcal{F}\mathsf{in}_* \to \mathcal{F}\mathsf{in}$ corresponds to polynomials whose middle map has finite fibers.

What's the polynomial functor of the bottom thing?

$$X \mapsto (X \mid F \in \mathcal{F}\mathsf{in}, p \in F) \mapsto (\prod_{p\in F} X \mid X \in \mathcal{F}\mathsf{in}) \mapsto \sum_{X\in\mathcal{F}\mathsf{in}} \prod_{p\in F} X.$$

Then

$$\sum_{X\in\mathcal{F}\mathsf{in}} \prod_{p\in F} X = \sum_{n\in\mathbb{N}} \sum_{p\in B\Sigma_n} X^n = \sum_{n\in\mathbb{N}} X^n /\!/ \Sigma_n =: e^X.$$

Note all polynomial functors are analytic functors: polynomial functors are much more general. Being analytic is a finiteness condition on polynomials (which is crazy).

Analytic functors. Recall that a polynomial

$$I \longleftarrow E \xrightarrow{p} B \longrightarrow J$$

is **analytic** if p has finite fibers which is equivalent to saying that this polynomial admits a morphism to e^X, the polynomial

$$* \longleftarrow \mathcal{F}\mathsf{in}_* \xrightarrow{u} \mathcal{F}\mathsf{in} \longrightarrow *$$

So $\mathcal{A}\mathsf{nFun} = \mathcal{P}\mathsf{olyFun}_{/\exp}$ is an ∞-topos.

Can we characterize analytic functors intrinsically?

defn. $X \in \mathcal{S}$ is **compact** if $\mathcal{S}(X,-)$ preserves filtered colimits and **projective** if $\mathcal{S}(X,-)$ preserves geometric realizations. $X \in \mathcal{S}$ is **finite** if $\mathcal{S}(X,-)$ is compact and also projective, which is the case iff $\mathcal{S}(X,-)$ preserves sifted colimits.

A proof of that equivalence is in [L1]. Note $X \in \mathcal{S}$ is finite iff X is a finite set.

Lemma. *$f\colon X \to Y$ is finite in $\mathcal{S}_{/Y}$ iff X is finite in $\mathcal{S}$.*

Proof. For any diagram $p\colon J \to \mathcal{S}_{/Y}$ we have a commutative diagram

$$\begin{tikzcd} \mathrm{colim}\,\mathcal{S}_{/Y}(X,p) \arrow[r] \arrow[d] & \mathcal{S}_{/Y}(X,\mathrm{colim}\,p) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \{f\} \arrow[d] \\ \mathrm{colim}\,\mathcal{S}(X,p) \arrow[r] & \mathcal{S}(X,\mathrm{colim}\,p) \arrow[r] & \mathcal{S}(X,Y) \end{tikzcd}$$

The right-hand square is a pullback by definition and the outer rectangle is also since colimits in $\mathcal{S}$ are universal (commuting with pullback, since pullback has a right adjoint). So the left-hand square is a pullback and in particular if the lower left horizontal is an equivalence the upper left one is as well.

For the other direction of the implication note

$$\mathcal{S}(X,\mathrm{colim}\,p) = \mathcal{S}_{/Y}(X, Y \times \mathrm{colim}\,p) \cong \mathcal{S}_{/Y}(X, \mathrm{colim}(Y \times p)).$$

□

Lemma. *If we have a span*

$$I \xleftarrow{f} X \xrightarrow{q} *$$

then $q_* f^*$ preserves sifted colimits iff X is finite.

So finally:

Proposition. *$F\colon \mathcal{S}_{/I} \to \mathcal{S}_{/J}$ is analytic iff F preserves weakly contractible limits and sifted colimits.*

NB: the paper does this the other way around, taking the above as a definition and deducing the characterization as polynomials for which p has finite fibers.

Recall from the pretalk that a stuff type is $F\colon \mathcal{F}\mathrm{in} \to \mathcal{S}$ (see page 27) and the induced analytic functor is defined by taking the left Kan extension. I can think of the stuff type as a symmetric sequence of homotopy types — the homotopy invariant notion of symmetric sequence.

This connects to material from Joachim Kock's previous paper [Koc].

defn. A **tree** is a diagram of finite sets

$$A \xleftarrow{s} M \xrightarrow{p} N \xrightarrow{t} A$$

so that t is injective, s is injective with a unique element $R \in A$ not in its image, and if we define $\sigma\colon A \to A$ by $\sigma(R) = R$ and for $e \in s(M)$ then $\sigma(e) = tps^{-1}(e)$ then for all e there exists $k \in \mathbb{N}$, $\sigma^k(e) = R$.

Why is this a tree? Think of A as a set of arcs, N is the set of notes, and M is the set of nodes paired with one of their input arcs. Then s and p are the projections, which t sends each node to its unique output. The root is the element R. The function σ walks to the root (along arcs, passing at each step through one node).

ex (elementary trees). For a tree with a single arc but no nodes you have

$$* \xleftarrow{s} \varnothing \xrightarrow{p} \varnothing \xrightarrow{t} *$$

which we call η. For the n-corolla

$$n+1 \xleftarrow{s} n \xrightarrow{!} 1 \xrightarrow{1} n+1$$

which we call C_n

Why do we care about these trees? They represent important analytic functors. Note that the hom-space $\mathcal{A}\mathsf{nEnd}_{/\exp}(\eta, P)$ between

$$\begin{tikzcd} * \arrow[d] & \varnothing \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \varnothing \arrow[r] \arrow[d] & * \arrow[d] \\ I & E \arrow[l] \arrow[r, "p"] & B \arrow[r] & I \end{tikzcd}$$

recovers the space I; note that in the category of analytic endofunctors you require the outside maps $* \to I$ to be the same.

Similarly, you only get a map

$$\begin{tikzcd} n+1 \arrow[d] & n \arrow[l] \arrow[r, "!"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[r] \arrow[d, "b"] & n+1 \arrow[d] \\ I & E \arrow[l] \arrow[r, "p"] & B \arrow[r] & I \end{tikzcd}$$

if p has n-elements in the fiber over b. Here the outside maps are determined.

Note you have maps $\eta \to C_n$ which pick out the colors or sorts in the set $n+1$ of arcs. There are maps $C_n \to C_n$ which permute the n elements.

By the first mapping space thing, we think of η as the homotopy type of colors, while C_n is the homotopy type of n-ary operations. The category of these elementary trees is called Ω_{El}. Without η, $\Omega_{\mathrm{El}} = \mathcal{F}\mathsf{in}$ but we throw in this additional object and some maps.

Theorem. *The restricted Yoneda embedding defines an equivalence*

$$\mathcal{A}\mathsf{nEnd} \simeq \mathcal{P}\mathsf{olyEnd}_{/\exp} \xrightarrow{\sim} \mathsf{Psh}(\Omega_{El}).$$

In particular, a map between analytic functors is an equivalence iff it looks like it when mapping out of η or the C_n.

This is getting very close to operads since the presheaf representation identifies the colors, maps $\eta \to F$, and the n-ary operations, maps $C_n \to F$, for all n.

Note we can compose analytic endofunctors.

There's a larger category of trees Ω_{int} built by gluing together the elementary trees in Ω_{el}.[6] Dendroidal sets are presheaves on this. Then Segal presheaves satisfy a further condition.

4. Initial Algebras and Free Monads — Naruki Masuda

In the theory of classical operads you start with a symmetric sequence $\{M(n)\}$. From this you can construct a "Schur functor" $X \mapsto \coprod M(n) \otimes X^{\otimes n}_{/\Sigma_n}$. In our language this is an analytic endofunctor.

[6]The "int" refers to inert maps of the inert-active factorization. If maps of trees are like diagrams of polynomials (with the middle map a pullback) you get an embedding of trees, I think.

Now an operad structure on a symmetric sequence corresponds to a monad structure on this endofunctor. There is a free operad construction, producing the free operad on a symmetric sequence, using trees of generating operations. Today we'll discuss this analog at the level of of endofunctors, building the free monad on an endofunctor.

Another classical tool from operad theory is the bar - cobar adjunction. The transposing arrows under these adjoints can be modeled more elementarily as twisting morphisms, giving a natural isomorphism:

$$\mathrm{Hom}_{\mathrm{alg}}(\Omega C, A) \simeq \mathrm{Tw}(C, A) \simeq \mathrm{Hom}_{\mathrm{coalg}}(C, BA).$$

Our aim today is to develop this in the ∞-categorical setting.

Lambek algebras. Let $\mathcal{C}$ be an ∞-category and let $P \in \mathrm{End}(\mathcal{C})$. Then a Lambek P-algebra is given by $\mu\colon PA \to A$ in $\mathcal{C}$. Lambek P-coalgebras are defined dually by $\delta\colon C \to PC$. The corresponding ∞-categories are defined by pullback

$$\begin{tikzcd} \mathrm{alg}_P\mathcal{C} \arrow[r] \arrow[d, two heads, "U"'] \arrow[dr, phantom, "\lrcorner" very near start] & \mathcal{C}^{\mathbb{2}} \arrow[d, two heads, "(\mathrm{dom},\mathrm{cod})"] \\ \mathcal{C} \arrow[r, "P\times\mathrm{id}"] & \mathcal{C}\times\mathcal{C} \end{tikzcd}$$

The ∞-category $\mathrm{coalg}_P\mathcal{C}$ is defined similarly.

Assume $\mathcal{C}$ is a category with filtered colimits and P preserves them.

Goal. If furthermore $\mathcal{C}$ has coproducts then U is monadic. if $\bar{P}$ is the corresponding monad then we get an equivalence of ∞-categories $\mathrm{alg}_P\mathcal{C} \simeq \mathrm{Alg}_{\bar{P}}\mathcal{C}$ over $\mathcal{C}$.

To construct the left adjoint F, recall $F \dashv U$ iff F is the absolute right Kan extension of the identity along U.[7] We can define the value of F at $x \in \mathcal{C}$ as the limit of the diagram indexed by a comma ∞-category:

$$\begin{tikzcd} {}^{X/}\mathrm{alg}_P\mathcal{C} \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow"] & \mathrm{alg}_P\mathcal{C} \arrow[r, equal] \arrow[d, "U"'] & \mathrm{alg}_P\mathcal{C} \\ \Delta[0] \arrow[r, "X"'] & \mathcal{C} \arrow[ur, dashed, "F"'] \arrow[u, phantom, shift left=4, "\Uparrow"] \end{tikzcd}$$

If ${}^{X/}\mathrm{alg}_P\mathcal{C}$ has an initial object then the limit exists and FX is defined by evaluation at this initial object.

defn. If $\mathcal{C}$ has binary coproducts, then we have a composite functor

$${}^{X/}\mathcal{C} \xrightarrow{\text{forget}} \mathcal{C} \xrightarrow{P} \mathcal{C} \xrightarrow{X \coprod -} {}^{X/}\mathcal{C}$$

This functor sends $X \to Y$ to $X \to X \coprod PY$ (forgetting the map).

Note that ${}^{X/}\mathcal{C}$ has filtered colimits and P_X preserves them (as a composite of functors that do with a left adjoint). Note also that $\mathrm{id}_X \in {}^{X/}\mathcal{C}$ is initial.

An object in alg_{P_X} is a pair $(X \to Y, PY \to Y)$, so this is equivalent to the comma ∞-category ${}^{X/}\mathrm{alg}_P\mathcal{C}$ constructed above.

It suffices to construct an initial P_X-algebra under these conditions, but we'll postpone it for now, since this uses bar-cobar duality and twisting morphisms.

[7] This is true in any 2-category; in particular, in the 2-category of ∞-categories, ∞-functors, and ∞-natural transformations.

Monadicity.

Theorem (Barr-Beck-Lurie [L2, 4.7.3.5]). *An adjunction $F \dashv U\colon \mathrm{alg}_P\mathcal{C} \to \mathcal{C}$ is monadic iff*

(i) U is conservative and
(ii) alg_P has colimits of U-split simplicial diagrams and U preserves them.

What U-split means is that your simplicial object extends as indicated by the dashed arrow

$$\begin{tikzcd} \mathbf{\Delta}^{\mathrm{op}} \arrow[r, "A"] \arrow[d, hook] & \mathrm{alg}_P\mathcal{C} \arrow[dd, "U"] \\ \mathbf{\Delta}_+^{\mathrm{op}} \cong (\mathbf{\Delta}^{\mathrm{op}})^{\triangleright} \arrow[ur, dotted] \arrow[d, hook] & \\ \mathbf{\Delta}_\perp^{\mathrm{op}} \arrow[r, dashed] & \mathcal{C} \end{tikzcd}$$

The image of $\mathbf{\Delta}_+^{\mathrm{op}} \hookrightarrow \mathbf{\Delta}_\perp^{\mathrm{op}}$ is an absolute colimit cone, so this gives a colimit for UA in $\mathcal{C}$. To say that $\mathrm{alg}_P\mathcal{C}$ has and U preserves these colimits is to say that this colimit cone lifts to a colimit cone in $\mathrm{alg}_P\mathcal{C}$.

Conservativity of U is easy to verify: if $f\colon X \to Y$ has an equivalence inverse $g\colon Y \to X$ this lifts to algebras.

To see that U-split colimits are created consider the defining pullback

$$\begin{tikzcd} \mathrm{alg}_P\mathcal{C} \arrow[r] \arrow[d, two heads, "U"'] \arrow[dr, phantom, "\lrcorner"] & \mathcal{C}^{\mathbb{2}} \arrow[d, two heads, "(\mathrm{dom},\mathrm{cod})"] \\ \mathcal{C} \arrow[r, "P\times \mathrm{id}"] & \mathcal{C}\times\mathcal{C} \end{tikzcd}$$

By co/monadicity, $\mathcal{C}^{\Delta[1]} \to \mathcal{C}\times\mathcal{C}$ preserves and reflects colimits. Since UA is absolute, P preserves it, as does the identity functor. So now the pullback ∞-category possesses and the functors preserve the colimit of $A\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{alg}_P\mathcal{C}$.

Twisting morphisms.

Goal. For a category $\mathcal{C}$ with filtered colimits and an endofunctor P that preserevs them then there exists a functor $\Omega = \Omega_P\colon \mathrm{coalg}_P \to \mathrm{alg}_P$. If dually $\mathcal{C}$ has cofiltered limits and P preserves them, then there exists a functor $B = B_P\colon \mathrm{alg}_P \to \mathrm{coalg}_P$ so that

$$\mathrm{Hom}_{\mathrm{alg}}(\Omega C, A) \simeq \mathrm{Tw}(C, A) \simeq \mathrm{Hom}_{\mathrm{coalg}}(C, BA).$$

To start, what is $\mathrm{Tw}(C, A)$?

For $\mathcal{C}$ an ∞-category, the twisted arrow ∞-category is characterized by the pullback

$$\begin{tikzcd} \mathrm{Tw}\mathcal{C} \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner"] & \mathcal{S}_* \arrow[d, "U"] \\ \mathcal{C}^{\mathrm{op}}\times\mathcal{C} \arrow[r, "\mathrm{Map}(-,-)"] & \mathcal{S} \end{tikzcd}$$

where U is the universal left fibration.

So an object in $\mathrm{Tw}\mathcal{C}$ is a morphism $f\colon X \to Y$ in $\mathcal{C}$. A morphism from f to $f'\colon X' \to Y'$ is given by a pair of maps $x\colon X' \to X$ and $y\colon Y \to Y'$ so that $f' = y \circ f \circ x$.

Note $P\mathrm{op} \times P$ acts on $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$. Form the pullback

$$\begin{tikzcd} \mathrm{Tw}_P(C,A) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{alg}_{\mathrm{Tw}P} \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & (\mathrm{Tw}\mathcal{C})^2 \arrow[d, two heads] \\ \Delta[0] \arrow[r, "{(C,A)}"'] & (\mathrm{coalg}_P)^{\mathrm{op}} \times \mathrm{alg}_P \simeq \mathrm{alg}_{P^{\mathrm{op}} \times P} \arrow[r] & (\mathcal{C}^{\mathrm{op}} \times \mathcal{C})^2 \end{tikzcd}$$

Explicitly, a twisting morphism $f \in \mathrm{Tw}_P(C,A)$ is $f\colon C \to A$ so that

$$\begin{tikzcd} PC \arrow[d, "Pf"'] \arrow[dr, phantom, "\simeq"] & C \arrow[l, "\delta"'] \arrow[d, "f"] \\ PA \arrow[r, "\mu"'] & A \end{tikzcd}$$

This can also be expressed as an equalizer

$$\mathrm{Tw}_P(C,A) \longrightarrow \mathrm{Map}_{\mathcal{C}}(C,A) \rightrightarrows \mathrm{Map}_{\mathcal{C}}(C,A)$$

of the identity with $\mu \circ - \circ \delta$.

Remark. Classically if P is a dg-algebra and C is a dg-coalgebra then $f\colon C \to P$ is a twisting morphism if when you define the convolution product $f \star f := \mu \circ f \otimes f \circ \delta$ then the **Maurer-Cartan** equation is satisfied $f \star f + \partial f = 0$.

Cobar construction. The functor $\mathrm{Tw}_P(-,A)\colon \mathcal{C}\mathrm{op} \to \mathcal{S}$ is representable. For a P-coalgebra $\delta\colon C \to PC$ define

$$P^\infty C := \mathrm{colim}_{n\to\infty}(C \xrightarrow{\delta} PC \xrightarrow{P\delta} P^2C \xrightarrow{P^2\delta} \cdots)$$

Since P preserves filtered colimits, $P^\infty C \simeq P \circ P^\infty C$ so this defines both a P-coalgebra $u\colon P^\infty C \to PP^\infty C$ and a P-algebra $v\colon PP^\infty C \to P^\infty C$. Denote this P-algebra by $\Omega C \in \mathrm{alg}_P\mathcal{C}$.

Proposition. *For all* $(A,\mu) \in \mathrm{alg}_P\mathcal{C}$, $\mathrm{Tw}_P(C,A) \simeq \mathrm{Map}_{\mathrm{alg}}(\Omega C, A)$.

Proof. Recall the equalizer

$$\begin{tikzcd} \mathrm{Tw}_P(P^\infty C,A) \arrow[r] \arrow[d, dashed, "\simeq"] & \mathrm{Map}_{\mathcal{C}}(P^\infty C,A) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, phantom, "\wr"] & \mathrm{Map}_{\mathcal{C}}(P^\infty C,A) \arrow[d, phantom, "\wr"] \\ \lim_{n\to\infty} \mathrm{Tw}_P(P^nC,A) \arrow[r] & \lim_{n\to\infty} \mathrm{Map}_{\mathcal{C}}(P^nC,A) \arrow[r, shift left] \arrow[r, shift right] & \lim_{n\to\infty} \mathrm{Map}_{\mathcal{C}}(P^nC,A) \end{tikzcd}$$

Commuting limits we get this dashed equivalence. So you just need to show that there is an equivalence $\lim_{n\to\infty} \mathrm{Tw}_P(P^nC,A) \simeq \mathrm{Tw}_P(C,A)$.

Note also that if we have inverse equivalences $u\colon U \to PU$ and $v\colon PU \to U$ then $\mathrm{Map}_{\mathrm{alg}}(U,A) \simeq \mathrm{Tw}_P(U,A)$ by comparing commutative squares. For the final step, the map $\delta^*\colon \mathrm{Tw}_P(PC,A) \to \mathrm{Tw}_P(C,A)$ given by $g \mapsto g \circ \delta$ is an equivalence with inverse given by $f \mapsto \mu \circ Pf$. □

Finally if $\mathcal{C}$ has an initial object $\varnothing$, then from this adjunction we have

$$* \simeq \mathrm{Map}_{\mathcal{C}}(\varnothing, A) \simeq \mathrm{Tw}_P(\varnothing, A) \simeq \mathrm{Map}_{\mathrm{alg}}(\Omega\varnothing, A)$$

so $\Omega\varnothing$ is the initial P-algebra that we wanted to construct.

Free monads. Now, if $\mathcal{C}$ has and P preserves filtered colimits, we have a monadic adjunction

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathrm{alg}_P\mathcal{C} \arrow[l, bend left, "U"] \end{tikzcd}$$

Write $\bar{P}$ for the induced monad. Monadicity gives an equivalence $\mathrm{alg}_P\mathcal{C} \simeq \mathrm{Alg}_{\bar{P}}\mathcal{C}$.

We claim that

Proposition. *$\bar{P}$ is a free monad on $\mathcal{C}$: i.e., $P \mapsto \bar{P}$ constructs a left adjoint to the forgetful functor from finitary monads on $\mathcal{C}$ to finitary endofunctors of $\mathcal{C}$.*

Proof. We want to show that $\bar{P}$ is the initial monad with a natural transformation $P \to \bar{P}$. Let $T = RL$ be the monad associated to a monadic adjunction

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathrm{Alg}_T\mathcal{C} \arrow[l, bend left, "R"] \end{tikzcd}$$

Then

$$\mathrm{Map}_{\mathrm{Mnd}(\mathcal{C})}(\bar{P}, T) \simeq \mathrm{Map}_{\mathsf{Cat}_\infty/\mathcal{C}}(\mathrm{Alg}_T, \mathrm{Alg}_{\bar{P}}) \simeq \mathrm{Map}_{\mathsf{Cat}_\infty/\mathcal{C}}(\mathrm{Alg}_T, \mathrm{alg}_P).$$

Since alg_P is defined by a pullback,

$$\begin{aligned} \mathrm{Map}_{\mathsf{Cat}_\infty/\mathcal{C}}(\mathrm{Alg}_T, \mathrm{alg}_P) &\simeq \mathrm{Map}_{\mathcal{C}\times\mathcal{C}}((PR, R)\colon \mathrm{Alg}_T \to \mathcal{C}\times\mathcal{C}, \mathcal{C}^2 \to \mathcal{C}\times\mathcal{C}) \\ &\simeq \mathrm{Nat}(PR, R) \simeq \mathrm{Map}_{\mathrm{End}(\mathcal{C})}(P, RL = T). \end{aligned}$$

□

Recall that for $X \in \mathcal{C}$,

$$\bar{P}(X) \simeq U \lim({}^{X/}\mathrm{alg}_P \to \mathrm{alg}_P),$$

and this limit is computed by evaluating at the initial object of ${}^{X/}\mathrm{alg}_P$, which is constructed by applying the cobar construction Ω_{P_X} to the initial object $X = X$ in ${}^{X/}\mathcal{C}$.

Explicitly,

$$\Omega_{P_X}(X = X) = \mathsf{colim}(X \hookrightarrow X \coprod PX \hookrightarrow X \coprod P(X \coprod PX) \hookrightarrow \cdots).$$

defn. Define $P_0 = \mathrm{id}$ and inductively define $P_{n+1} = \mathrm{id} \coprod P \circ P_n$ together with natural transformations $f_0\colon P_0 \to P_1 = \mathrm{id} \coprod P$ given by inclusion into the first component and $f_{n+1} = \mathrm{id} \coprod P(f_n)$. Then

$$\bar{P}X \simeq (\mathsf{colim} P_n)X.$$

Remark. The free operad construction from a symmetric sequence $\{M(n)\}$ is $\{TM(n)\}$ where $TM(n)$ is trees with n leaves with nodes labeled by operations in M of the appropriate arity.

In the context of symmetric sequences the identity corresponds to the symmetric sequence $\{I(n)\}$ which is 1 if $n = 1$ and 0 otherwise.

Then

$$TM \coloneqq \mathsf{colim}(I \hookrightarrow I \coprod M \hookrightarrow I \coprod M \circ (I \coprod M) \hookrightarrow \cdots)$$

and each stage of this colimit adds trees of height at most n.

So these explicit constructions are totally analogous.

Note this gives an equivalence of endofunctors but not yet an equivalence of monads. You need to put a monoid structure on the colimits. This can be done in a straightforward way by taking colimits of the composition $\mu_{n,m}\colon P_n \circ P_m \to P_{n+m}$. These maps can again be defined recursively:

- $\mu_{0,m}\colon \mathrm{id}\circ P_m = P_m$
- $\mu_{n+1,m}\colon P_{n+1}\circ P_m = P_m \coprod P\circ P_n\circ P_m \xrightarrow{\mathrm{id}\coprod P\mu_{n,m}} P_m \coprod P\circ P_{n+m} \hookrightarrow P_{n+m+1}$

Since P commutes with filtered colimits so does P_n so we can compute the colimits in any order.

Parametrized version. When $\mathcal{C}$ has sifted colimits and P preserves them and $\mathcal{C}$ has coproduct, then the free monad on an endofunctor adjunction restricts to

$$\begin{tikzcd} \mathrm{End}^{\sigma}(\mathcal{C}) \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathrm{Mnd}^{\sigma}(\mathcal{C}) \arrow[l, bend left, "U"] \end{tikzcd}$$

where these are the ∞-categories of ∞-categories with and functors preserving sifted colimits.

This adjunction in fact is monadic. Then $\mathrm{End}^{\sigma}(\mathcal{C})$ is presentable at least when $\mathcal{C}$ is **sifted presentable**. This means that there exists a small ∞-category $\mathcal{C}_0$ with coproducts so that $\mathcal{C}$ is equivalent to $P_{\Sigma}(\mathcal{C}_0)$, presheaves that carry (finite?) coproducts to products. So then $\mathrm{Mnd}^{\sigma}(\mathcal{C})$ is again presentable.

What we want is a free operad construction

$$\begin{tikzcd} \mathcal{A}\mathsf{nMnd} \arrow[rr] \arrow[dr] & & \mathcal{A}\mathsf{nEnd} \arrow[ll, dashed, bend right, "\perp"'] \arrow[dl] \\ & \mathcal{S} & \end{tikzcd}$$

which fiberwise would have the form

$$\begin{tikzcd} \mathcal{A}\mathsf{nMnd}(I) \arrow[r] & \mathcal{A}\mathsf{nEnd}(I) \arrow[l, dashed, bend right, "\perp"'] \end{tikzcd}$$

where the left-hand side corresponds to "operads" and the right-hand side corresponds to "symmetric sequences."

This will be obtained by pulling back something very complicated to obtain

$$\begin{tikzcd} \mathrm{Mnd}^{\mathrm{colax,op}} \arrow[rr] \arrow[dr] & & \mathrm{End}^{\mathrm{colax,op}} \arrow[dl] \\ & \mathsf{Cat}_{\infty}^{\mathrm{op}} & \end{tikzcd}$$

Fiberwise over $\mathcal{C}\in\mathsf{Cat}_\infty$ this gives $\mathrm{Mnd}(\mathcal{C})$ and $\mathrm{End}(\mathcal{C})$. This functor over $\mathsf{Cat}_\infty^{\mathrm{op}}$ is constructed by a universal property, so it seems reasonable that it would correspond fiberwise to the forgetful functor $\mathrm{Mnd}(\mathcal{C})\to\mathrm{End}(\mathcal{C})$ but the authors don't verify this (see Warning B.3.1).

Theorem. *The forgetful functor* $\mathrm{Mnd}^{colax,\mathrm{op}}\to\mathrm{End}^{colax,\mathrm{op}}$ *has a left adjoint and is monadic.*

5. Analytic Monads and ∞-Operads — Daniel Fuentes-Keuthan

A combinatorial model for ∞-operads. The structure of a category — objects, arrows, compositions of arrows — is controlled by the category $\mathbf{\Delta}$. One way this manifests is in the fully faithful nerve functor $N\colon \mathsf{Cat}\to\mathsf{sSet}$. Also

Proposition. *If X is a simplicial set that admits unique fillers*

$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d, hook] & X \\ \Delta[n] \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

for $n \geq 2$, $0 < k < n$, then X is isomorphic to the nerve of a category.

This also gives a nice model for $(\infty, 1)$-categories: just relax the uniqueness to existence.

By analogy we can consider symmetric multicategories, which nowadays everyone just calls (colored) **operads**, which have objects, arrows with n-ary source and unary target, and multicomposition. Categories embed into operads so we might ask

$$\begin{tikzcd} \mathsf{Cat} \arrow[r, hook] \arrow[d, hook] & \mathsf{sSet} \arrow[d, hook] \\ \mathsf{Operad} \arrow[r, hook] & ?? \end{tikzcd}$$

The ?? is filled by the category of **dendroidal sets**, which are presheaves on the tree category $\mathbf{\Omega}$.

The tree category. Why trees? Objects in $\mathbf{\Delta}$ are finite linear orders, which is good for composition of unary arrows but not sufficient for composition of multiarrows.

Everyone has different trees. Our trees with have:

- a **root** labelling the bottom edge
- a **root note** labelling the last composition
- **internal edges**
- **internal nodes**
- **leaf nodes** (for nullary composition)
- **leaf edges** (for the source objects)

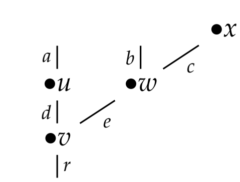
Remark. Unfortunately when you draw a tree you fix a planar structure but importantly the trees that arise here *are not planar*. This is actually the source of a lot of the complexity: trees have non-trivial automorphisms.

Note that a morphism $[n] \to [m]$ in $\mathbf{\Delta}$ is a functor $[n] \to [m]$ from the free categories on the directed graphs. We'll define morphisms on trees similarly.

defn. The **free operad** on a tree T has

- objects the edges of T
- morphisms generated by the vertices

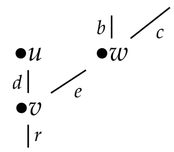
ex. For the tree

$T =$ 

$$\Omega[T](d, e; r) = \{v\},$$
$$\Omega[T](a; d) = \{u\},$$
$$\Omega[T](d, b, c; r) = \{v \circ_e w\},$$

etc.

Morphisms in $\mathbf{\Omega}$ will be morphisms between free operads generated by trees. So for example, there's a morphism from the 2-corolla to the tree



that sends the binary operation to $v \circ (u, w)$. Note, however, that the are no morphisms from the 2-corolla to the 3-corolla because the arities don't match.

ex. The free operad on the tree

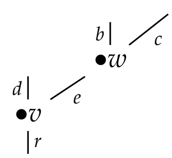
$L_2 :=$ 

is the category $[2]$. This defines the embedding $\mathbf{\Delta} \hookrightarrow \mathbf{\Omega}$.

Note the only trees that admit maps to $L_0 =: \eta$ are the linear trees. So $\mathbf{\Omega}_{/\eta} \cong \mathbf{\Delta}$.

Morphisms come in different kinds. **Outer face maps** include

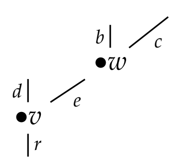
- **leaf faces** that trim the tree



  at the vertex w, mapping a 2-corolla to the vertex v
- **root faces** that trim the root, mapping a 2-corolla to the vertex v.

Then there are **inner face maps** which include

- **edge collapse**, which map from the 3-corolla to the "composite" arrow



Finally there are **degeneracies** which map some linear subtree to a root (sending unary edges to identities).

Proposition.

(i) *The category $\mathbf{\Omega}$ can be described via the codendroidal identities.*
(ii) *Every morphism factors as faces, followed by symmetries, followed by degeneracies.*
(iii) *Every composite of faces can be written as a composite of inner faces followed by outer faces or as a composite of outer faces followed by inner faces.*
(iv) *Every morphisms factors as surjective on edges and vertices followed by injective on edges.*

Dendroidal sets.

defn. A **dendroidal set** is a presheaf on $\mathbf{\Omega}$.

Proposition. *We can recover* $\mathsf{sSet} \cong \mathsf{dSet}_{/\Omega[\eta]}$. *This also describes the image of the left Kan extension under the adjunction*

$$\begin{tikzcd} \mathsf{sSet} \arrow[r, bend left, "i_!"] \arrow[r, phantom, "\perp"] & \mathsf{dSet} \arrow[l, bend left, "i^*"] \end{tikzcd}$$

induced by $i \colon \mathbf{\Delta} \hookrightarrow \mathbf{\Omega}$.

Proposition. *There is a fully faithful inclusion* $N_d \colon \mathsf{Operad} \hookrightarrow \mathsf{dSet}$ *defined by*

$$N_d(O)(T) = \mathsf{Operad}(\Omega[T], O).$$

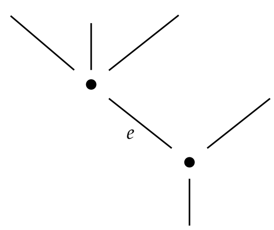
This functor also admits a left adjoint, defining the "homotopy multicategory" of a dendroidal set.

We can characterize the essential image of the dendroidal nerve.

defn. An **inner horn** of a tree T at an inner edge e is the subobject of all of the outer and inner face maps of T except for one: the inner face map defined by edge collapse at the edge e.

ex. The inner horn of L_2 is the union of what you get when you prune away the root and when you prune away the leaf. This corresponds to $L_1 \cup_{L_0} L_1$, which is the usual inner horn of the 2-simplex.

ex. Exercise is to work out the inner horn opposite the edge e for



This includes the two leaf face maps, the root face map, and the edge collapse map at the other inner edge.

Proposition. *A dendroidal set X is in the essential image of N_d if and only if every inner horn admits a unique filler.*

Q. Which dendroidal sets represent symmetric monoidal categories?

defn. A **quasi-operad** is a dendroidal set such that every horn admits a filler.

The **Boardman Vogt** W- construction has the form of a functor $W \colon \mathbf{\Omega} \to \mathsf{sOp}$ that sends T to $W\Omega[T]$, which is some sort of simplicial resolution of the free operad $\Omega[T]$. This induces an homotopy coherent nerve type adjunction

$$\begin{tikzcd} \mathsf{dSet} \arrow[r, bend left, "C"] \arrow[r, phantom, "\perp"] & \mathsf{sOp} \arrow[l, bend left, "N"] \end{tikzcd}$$

If a simplicial operad O is enriched in Kan complexes then its homotopy coherent nerve is a quasi-operad.

There is a (non-Cisinski) model structure due to Cisinski-Moerdijk.

Theorem (Cisinski-Moerdijk)**.** *There is a model structure on* dSets.

Note not all monomorphisms are cofibrations. In particular, not all objects are cofibrant.

Theorem (Barwick-Chu-Cisinski-Haugseng-Heuts-Hinich-Lurie-Moerdijk)**.** *There are equivalences of homotopy theories*

$$\mathsf{dSet} \simeq \mathsf{sOp} \cong \mathsf{Lurie\ operads} \simeq \mathsf{SegOp} \simeq \mathsf{cdsSet}.$$

Moreover

Theorem (Cisinski-Moerdijk)**.** *There is a Quillen equivalence* $\mathsf{dSet}_{/\eta} \simeq \mathsf{sSet}$ *with the Joyal model structure.*

The model used in this paper is the **complete dendroidal Segal spaces**.

Theorem (Cisinski-Moerdijk)**.** *There are commuting Quillen equivalences*

$$\begin{array}{ccc} \mathsf{dSet}_{/\eta} & \simeq & \mathsf{cdsSet}_{/\eta} \\ \simeq & & \simeq \\ \mathsf{sSet} & \simeq & \mathsf{CSS} \end{array}$$

As an aside:

Theorem (Heuts)**.** *Dendroidal sets with the right lifting property with respect to inner horns and leaf horns model* E_∞*-spaces.*

Theorem (Nikolaus)**.** *Dendroidal sets with the right lifting property with respect to all horns model connective spectra.*

Analytic monads as complete dendroidal Segal spaces. Note completeness isn't mentioned in this paper anywhere, so it's an exercise to understand the role played by that condition.

Let's review.

defn. An **analytic endofunctor** is a polynomial

$$\begin{tikzcd} I \arrow[d] & E \arrow[l] \arrow[r, "p"] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & B \arrow[r] \arrow[d] & I \arrow[d] \\ * & \mathcal{F}\mathsf{in}_* \arrow[l] \arrow[r] & \mathcal{F}\mathsf{in} \arrow[r] & * \end{tikzcd}$$

where the pullback imposes the condition that p has finite fibers.

defn. A **tree** is a polynomial

$$A \xleftarrow{\text{input}} M \xrightarrow{\text{node}} N \xrightarrow{\text{output}} A$$

so that

(i) the input map is injective and misses one object
(ii) the output is injective

and these objects are all finite sets. Here A is the set of arcs, N is the set of nodes, and M is the set of nodes paired with one of their input arcs (this misses the leaf nodes). These give trees of finite height.

The tree category $\mathbf{\Omega}$ discussed in the pretalk is a full subcategory of $\mathcal{A}\mathsf{nEnd}$ spanned by the trees. Since the objects of the tree category are sets, this sub ∞-category is a 1-category.

ex. η is the tree

$$* \longleftarrow \varnothing \longrightarrow \varnothing \longrightarrow *$$

The corolla C_n is the tree

$$n+1 \longleftarrow n \longrightarrow * \longrightarrow n+1$$

Lemma.

- $\mathcal{A}\mathsf{nEnd}(\eta, P) = I$, *the set of colors.*
- $\mathcal{A}\mathsf{nEnd}(C_n, P) = B_n$, *the set of operations of arity n.*

defn. The category $\mathbf{\Omega}_{\mathrm{el}} \subset \mathcal{A}\mathsf{nEnd}$ is the full sub ∞-category spanned by η and C_n.

defn. The category $\mathbf{\Omega}_{\mathrm{int}} \subset \mathcal{A}\mathsf{nEnd}$ is the full sub ∞-category of trees an **inert maps**.

Proposition. *The composite*

$$\mathcal{A}\mathsf{nEnd} \xrightarrow{\text{よ}} P(\mathcal{A}\mathsf{nEnd}) \longrightarrow P(\mathbf{\Omega}_{el})$$

is an equivalence.

Lemma. *The inclusion $\mathbf{\Omega}_{\mathsf{el}} \hookrightarrow \mathbf{\Omega}_{\mathsf{int}}$ induces a restriction $\dashv$ right Kan extension adjunction on presheaves.*

These results combine to identify $\mathcal{A}\mathsf{nEnd}$ with a sub ∞-category of $P(\mathbf{\Omega}_{\mathsf{int}})$, which we now describe. It's image is spanned by those $X \in P(\mathbf{\Omega}_{\mathsf{int}})$ so that the Segal condition holds, i.e.,

$$X(T) \xrightarrow{\sim} \lim_{E \in \mathsf{el}(T)} X(E)$$

is an equivalence. The limit correspond to building a tree by grafting corollas C_n along edges η. The conclusion is that $\mathcal{A}\mathsf{nEnd}$ is equivalent to the full sub ∞-category of Segal presheaves on $\mathbf{\Omega}_{\mathsf{int}}$.

We've seen that Segal presheaves on $\mathbf{\Omega}_{\mathsf{int}}$ correspond to analytic endofunctors. These are closely related to dendroidal Segal spaces.

defn. A **dendroidal Segal space** is a functor $X\colon \mathbf{\Omega}^{\mathrm{op}} \to \mathcal{S}$ so that

$$X(T) \xrightarrow{\sim} \lim_{E \in \mathsf{el}(T)} X(E).$$

That is dendroidal Segal spaces are Segal presheaves on $\mathbf{\Omega}$.

The difference is we've extended along the inclusion $\mathbf{\Omega}_{\mathsf{int}} \hookrightarrow \mathbf{\Omega}$. This corresponds to adding in the "active maps," which govern composition (and hence the operad structure). If Segal presheaves without this composition are analytic endofunctors we might expect that Segal presheaves on $\mathbf{\Omega}$ are analytic monads, and indeed we'll see that this is true.

Analytic monads. Recall that the inclusion of polynomial monads on I to polynomial endofunctors on I admits a left adjoint, forming the free polynomial monad on a polynomial endofunctor.

defn. An **analytic monad** is a polynomial monad whose underlying endofunctor is analytic.

Proposition. *The free polynomial monad on an analytic endofunctor is analytic.*

Proof. Let P be a polynomial endofunctor. Then $\bar{P}$ is the free monad defined as a colimit $\bar{P} = \mathsf{colim} P_n$ where $P_0 = \mathrm{id}$ and $P_{n+1} = \mathrm{id} \coprod P \circ P_n$. You prove inductively that the P_n's are analytic. You show also that the maps $P_n \to P_{n+1}$ are cartesian. It will follow that their colimit is as well. The maps on $\bar{P}$ are built out of cartesian maps $P_n \circ P_n \to P_{2n}$ and these are cartesian. □

When P is an analytic endofunctor, the free analytic monad $\bar{P}$ has a nice description.

defn. For a polynomial P, a P-tree is a map

$$\begin{tikzcd} A \arrow[d] & M \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & N \arrow[r] \arrow[d] & A \arrow[d] \\ I & E \arrow[l] \arrow[r, "p"] & B \arrow[r] & I \end{tikzcd}$$

which takes the tree data and labels the edges by the elements of I and labels the nodes by elements of B of appropriate arity.

The mapping spaces of P can be extracted as follows: define

$$\begin{tikzcd} P(c_1, \ldots, c_n; c) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{A}\mathsf{nEnd}(C_n, P) \simeq B_n \arrow[d] \\ * \arrow[r, "(c_1, \ldots, c_n, c)"] & \mathrm{Map}(n+1, I) \end{tikzcd}$$

defn. The ∞-category $\mathsf{tr}(P)$ is the ∞-category of P-trees. The ∞-category $\mathsf{tr}'(P)$ is the ∞-category of P-trees with a marked edge.

Proposition. *The polynomial*

$$I \longleftarrow \mathsf{tr}'(P) \longrightarrow \mathsf{tr}(P) \longrightarrow I$$

is equivalent to the free analytic monad on P.

Main theorem.

defn. Let $\mathbf{\Omega}$ denote the full sub ∞-category (secretly a 1-category) of $\mathcal{A}\mathsf{nMnd}$ spanned by the trees.

defn. A **Segal presheaf** on $\mathbf{\Omega}$ is a functor $X\colon \mathbf{\Omega}^{\mathrm{op}} \to \mathcal{S}$ so that the restriction $X\colon \mathbf{\Omega}_{\mathsf{int}}^{\mathrm{op}} \to \mathcal{S}$ is a Segal presheaf.

The ∞-category of Segal presheaves on $\mathbf{\Omega}$ is defined by the evident pullback

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & P(\mathbf{\Omega}) \arrow[d] \\ P_{\mathrm{Segal}}(\mathbf{\Omega}_{\mathsf{int}}) \arrow[r] & P(\mathbf{\Omega}_{\mathsf{int}}) \end{tikzcd}$$

We've finally reached the main theorem:

Theorem. *The restricted Yoneda embedding*

$$\mathcal{A}\mathsf{nMnd} \to P(\mathbf{\Omega})$$

is fully faithful with essential image the Segal presheaves.

The proof uses:

Proposition. *Given a diagram of ∞-categories*

$$\begin{tikzcd} \mathcal{E}_1 \arrow[r, "\bar{\phi}"] \arrow[d, "u_1"'] & \mathcal{E}_2 \arrow[d, "u_2"] \\ \mathcal{B}_1 \arrow[r, "\phi"'] & \mathcal{B}_2 \end{tikzcd}$$

such that

- *u_i admits a left adjoint*
- *the adjunctions are monadic*
- *the mate $F_2\phi \Rightarrow \bar{\phi}F_1$ is an equivalence*
- *ϕ is fully faithful*

then $\bar{\phi}$ is fully faithful with essential image those $A \in \mathcal{E}_2$ so that $U_2A \in \mathrm{im}\phi$.

Proof of main theorem. We'll apply that result to the diagram

$$\begin{tikzcd} \mathcal{A}\mathsf{nMnd} \arrow[r, "\bar{\phi}"] \arrow[d, hook] & P(\mathbf{\Omega}) \arrow[d, "j^*"'] \\ \mathcal{A}\mathsf{nEnd} \arrow[r, "\phi"] & P(\mathbf{\Omega}_{\mathsf{int}}) \end{tikzcd}$$

We have the functors and the adjoints. We need to show that the mate is an equivalence: $j_!\phi \simeq \bar{\phi}F$. Since j^* is conservative it suffices to show that $j^*j_!\phi \simeq j^*\bar{\phi}F = \phi UF$ componentwise for each analytic endofunctor P. You prove this first pointwise for the elementary trees T and then apply the Segal condition to argue that this is enough. □

References

[Ber] J. E. Bergner. *A model category structure on the category of simplicial categories.* Trans. Amer. Math. Soc. 359 (2007), no. 5, 2043–2058.

[BV] J. M. BoardmanR. M. Vogt *Homotopy Invariant Algebraic Structures on Topological Spaces* Springer, Lecture Notes in Math, volume 347, 1973.

[Cis] Denis-Charles Cisinski *Higher categories and homotopical algebra* Cambridge University Press.

[GHK] David Gepner, Rune Haugseng, Joachim Kock *∞-Operads as Analytic Monads* arXiv:1712.06469.

[Joy] A. Joyal. *Quasi-categories and Kan complexes.* J. Pure Appl. Algebra 175 (2002), no. 1-3, 207–222.

[L1] J. Lurie. *Higher Topos Theory*. Annals of Mathematics Studies, 170, Princeton University Press, 2009.

[L2] J. Lurie. *Higher Algebra*

[Koc] J. Koch. *Polynomial functors and trees* International Mathematics Research Notices, Volume 2011, Issue 3, 2011, Pages 609–673.

[RV] E. Riehl and D. Verity *The comprehension construction*. Higher Structures, Vol 2, No 1 (2018).