HOMOTOPY COHERENT STRUCTURES

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Abstract. Naturally occurring diagrams in algebraic topology are commutative up to homotopy, but not on the nose. It was quickly realized that very little can be done with this information. Homotopy coherent category theory arose out of a desire to catalog the higher homotopical information required to restore constructibility (or more precisely, functoriality) in such "up to homotopy" settings. The first lecture will survey the classical theory of homotopy coherent diagrams of topological spaces. The second lecture will revisit the free resolutions used to define homotopy coherent diagrams and prove that they can also be understood as homotopy coherent realizations. This explains why diagrams valued in homotopy coherent nerves or more general ∞-categories are automatically homotopy coherent. The final lecture will venture into homotopy coherent algebra, connecting the newly discovered notion of homotopy coherent adjunction to the classical cobar and bar resolutions for homotopy coherent algebras.

Contents

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Part I. Homotopy coherent diagrams

I.1. Historical motivation

If X is a G-space, for G a discrete group, and Y is homotopy equivalent to X, then is Y a G-space? The action of a group element $g \in G$ on Y can be defined by transporting along the maps $f\colon X \to Y$ and $f^{-1}\colon Y \to X$ of the homotopy equivalence:

$$\begin{tikzcd} Y \arrow[r, "f^{-1}"] \arrow[d, dashed, "g_*"'] & X \arrow[d, "g_*"] \\ Y & X \arrow[l, "f"] \end{tikzcd}$$

This defines a continuous endomorphism of Y for every $g \in G$, as required by a G-action, but these maps are not necessarily automorphisms (since f and f^{-1} need not be homeomorphisms) nor is the composite of the actions associated to a pair of elements $g, h \in G$ equal to the action by their product: instead

$$g_* \circ h_* = (f \circ g_* \circ f^{-1}) \circ (f \circ h_* \circ f^{-1}) = f \circ g_* \circ (f^{-1} \circ f) \circ h_* \circ f^{-1} \quad \text{and}$$
$$(gh)_* = f \circ (gh)_* \circ f^{-1}$$

are homotopic via the homotopy $f^{-1} \circ f \simeq \mathrm{id}_X$.

So if Y is not a G-space, then what is it? The main aim of Part I is to develop language to describe this sort of situation. A G-space X may be productively considered as a *diagram* in the category of topological spaces indexed by a category $\mathsf{B}G$ with a single object and with an endomorphism corresponding to each element in the group.[1]

By contrast the "up to homotopy G-space" Y is instead a *homotopy commutative diagram*. Modulo point-set topological considerations that we sweep under the rug using a technique that will be described below, the category of topological spaces is self-enriched, meaning that the set of continuous functions between any pair of spaces X and Y is itself a space, which we denote by $\mathrm{Map}(X, Y)$. The points in $\mathrm{Map}(X, Y)$ are the continuous functions $f\colon X \to Y$ while a path in $\mathrm{Map}(X, Y)$ between points f and g is a **homotopy**. Two parallel maps $f, g\colon X \to Y$ are **homotopic** — in symbols "$f \simeq g$" — just when they are in the same path component in $\pi_0\mathrm{Map}(X, Y)$. Importantly, composition of continuous functions itself defines a continuous function between mapping spaces

$$\mathrm{Map}(Y, Z) \times \mathrm{Map}(X, Y) \xrightarrow{\circ} \mathrm{Map}(X, Z)$$

so this relation of taking homotopy classes of maps is preserved by pre- and post-composition with another continuous function. This defines the category $\mathsf{h}\mathbb{S}\mathsf{pace}$ of spaces and homotopy classes of continuous functions as a quotient of the enriched category $\mathbb{S}\mathsf{pace}$ of spaces and mapping spaces $\mathrm{Map}(X, Y)$, the points in which are

[1] Here "diagram" is synonymous with "functor": to define a **functor** whose domain is the category $\mathsf{B}G$ is to specify an image for the single object together with and endomorphism g_* for each $g \in G$ so that $(gh)_* = g_* h_*$ and $e_* = \mathrm{id}$.

continuous functions, the paths in which are homotopies, and the paths between paths in which are higher homotopies.

Definition I.1.1. If A is an ordinary category then

- a **diagram of spaces** is just a functor $\mathsf{A} \to \mathbb{S}\mathsf{pace}$
- a **homotopy commutative diagram of spaces** is a functor $\mathsf{A} \to \mathsf{h}\mathbb{S}\mathsf{pace}$

Thus, the G-space X defines a diagram $X \colon \mathsf{B}G \to \mathbb{S}\mathsf{pace}$ while the "up-to-homotopy" G-space Y defines a homotopy commutative diagram $Y \colon \mathsf{B}G \to \mathsf{h}\mathbb{S}\mathsf{pace}$. This terminology suggests a related question: is every "up-to-homotopy" G-space realized by a homotopy equivalent G-space? Or more generally, when does a homotopy commutative diagram $F \colon \mathsf{A} \to \mathsf{h}\mathbb{S}\mathsf{pace}$ admit a **realization**, i.e., a diagram $F' \colon \mathsf{A} \to \mathbb{S}\mathsf{pace}$ together with homotopy equivalences $Fa \simeq F'a$ that define a natural transformation[2] in $\mathsf{h}\mathbb{S}\mathsf{pace}$.

Theorem I.1.2 ([DKS, 2.5])**.** *A homotopy commutative diagram has a realization if and only if it may be lifted to a homotopy coherent diagram. Moreover, equivalence classes of realizations correspond bijectively with equivalences classes of homotopy coherent diagrams.*

Proof. This result is proven as a corollary of [DKS, 2.4] which demonstrates that appropriately defined spaces of homotopy coherent diagrams and realizations (defined slightly differently than above) are weak homotopy equivalent. □

In particular, since the homotopy commutative G-space Y is realized by the G-space X, it must underlie a homotopy coherent diagram of shape $\mathsf{B}G$. Our task is now to work out what exactly the phrase *homotopy coherent diagram* means.

I.2. The shape of a homotopy coherent diagram

To build intuition for the general notion of a homotopy coherent diagram, it is helpful to consider a special case. To that end, let

$$\omega := \qquad 0 \longrightarrow 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots$$

denote the category whose objects are finite ordinals and with a morphism $j \to k$ if and only if $j \leq k$.

A ω-shaped graph in $\mathbb{S}\mathsf{pace}$ is comprised of spaces X_k for each $k \in \omega$ together with morphisms $f_{j,k} \colon X_j \to X_k$ whenever $j < k$.[3] This data defines a *homotopy commutative diagram* $\omega \to \mathsf{h}\mathbb{S}\mathsf{pace}$ just when $f_{i,k} \simeq f_{j,k} \circ f_{i,j}$ whenever $i < j < k$.

To extend this data to a *homotopy coherent diagram* $\omega \to \mathbb{S}\mathsf{pace}$ requires:

- Chosen homotopies $h_{i,j,k} \colon f_{i,k} \simeq f_{j,k} \circ f_{i,j}$ whenever $i < j < k$. This amounts to specifying a path in $\mathrm{Map}(X_i, X_k)$ from the vertex $f_{i,k}$ to the vertex $f_{j,k} \circ f_{i,j}$, which is obtained as the composite of the two vertices $f_{i,j} \in \mathrm{Map}(X_i, X_j)$ and $f_{j,k} \in \mathrm{Map}(X_j, X_k)$.

[2] See Definition I.3.5.

[3] To simplify somewhat we adopt the convention that $f_{j,j}$ is the identity, making this data into a **reflexive directed graph** with implicitly designated identities.

- For $i < j < k < \ell$, the chosen homotopies provide four paths in $\mathrm{Map}(X_i, X_\ell)$

$$\begin{tikzcd} f_{i,\ell} \arrow[r, dash, "h_{i,k,\ell}"] \arrow[d, dash, "h_{i,j,\ell}"'] & f_{k,\ell} \circ f_{i,k} \arrow[d, dash, "f_{k,\ell} \circ h_{i,j,k}"] \\ f_{j,\ell} \circ f_{i,j} \arrow[r, dash, "h_{j,k,\ell} \circ f_{i,j}"'] & f_{k,\ell} \circ f_{j,k} \circ f_{i,j} \end{tikzcd}$$

  We then specify a higher homotopy — a *2-homotopy* — filling in this square.
- For $i < j < k < \ell < m$, the previous choices provide 12 paths and six 2-homotopies in $\mathrm{Map}(X_i, X_m)$ that assemble into the boundary of a cube. We then specify a *3-homotopy*, a homotopy between homotopies between homotopies, filling in this cube.
- Etc.

Even in this simple case of the category ω, this data is a bit unwieldy. Our task is to define a category to index this homotopy coherent data arising from ω: the objects X_i, the functions $X_i \to X_j$, the 1-homotopies $h_{i,j,k}$, the 2-homotopies, and so on. This data will assemble into a *simplicial category* whose objects are the same as the objects of ω but which will have n-morphisms in each dimension $n \geq 0$, to index the n-homotopies.

Because of the convenience of the mechanism of simplicial categories, and to avoid the point-set topology considerations alluded to above, we should now come clean and admit that we prefer to assume that our "mapping spaces" are really simplicial sets, or more precisely *Kan complexes* in the case of $\mathbb{S}\mathsf{pace}$ and similar categories, about more which below.

Digression I.2.1 (a crash course on simplicial sets). There is a convenient category whose objects model topological spaces (at least up to weak homotopy type): the category sSet of simplicial sets. A **simplicial set** X is a graded set $(X_n)_{n\geq 0}$ of n-simplices together with maps

$$X := \quad \begin{tikzcd} X_0 \arrow[r, tail] & X_1 \arrow[l, two heads, shift left=2] \arrow[l, two heads, shift right=2] \arrow[r, tail, shift left=1] \arrow[r, tail, shift right=1] & X_2 \arrow[l, two heads, shift left=2] \arrow[l, two heads] \arrow[l, two heads, shift right=2] \arrow[r, tail, shift left=2] \arrow[r, tail] \arrow[r, tail, shift right=2] & X_3 \arrow[l, two heads, shift left=3] \arrow[l, two heads, shift left=1] \arrow[l, two heads, shift right=1] \arrow[l, two heads, shift right=3] & \cdots \end{tikzcd} \tag{I.2.2}$$

that fulfill two functions:

- The $n+1$ **face maps** $X_n \twoheadrightarrow X_{n-1}$ identify the faces of an n-simplex.
- The n **degeneracy maps** $X_n \rightarrowtail X_{n+1}$ define degenerate $n+1$-simplices that project onto a given n-simplex.

There is a slick way to make all of this precise. Let $\mathbb{\Delta}$ denote the category of finite non-empty ordinals $[n] = \{0, 1 \ldots, n\}$ and order-preserving maps. The maps in the category $\mathbb{\Delta}$ are generated under composition by the basic inclusions and surjections displayed here:

$$\mathbb{\Delta} := \quad \begin{tikzcd} {[0]} \arrow[r, tail, shift left=2] \arrow[r, tail, shift right=2] & {[1]} \arrow[l, two heads] \arrow[r, tail, shift left=2] \arrow[r, tail] \arrow[r, tail, shift right=2] & {[2]} \arrow[l, two heads, shift left=1] \arrow[l, two heads, shift right=1] \arrow[r, tail, shift left=3] \arrow[r, tail, shift left=1] \arrow[r, tail, shift right=1] \arrow[r, tail, shift right=3] & {[3]} \arrow[l, two heads, shift left=2] \arrow[l, two heads] \arrow[l, two heads, shift right=2] & \cdots \end{tikzcd}$$

Now a **simplicial set** is just a contravariant functor $X \colon \mathbb{\Delta}^{\mathrm{op}} \to \mathsf{Set}$.

The category sSet is generated by the standard n-simplices Δ^n, which we think of geometrically as an (ordered) n-simplex spanned by the vertices $0,\ldots,n$. The standard n-simplex is the functor represented by the object $[n] \in \mathbb{\Delta}$. There are various maps between these standard simplices

$$\mathbb{\Delta} := \qquad \Delta^0 \;\substack{\rightarrowtail \\ \twoheadleftarrow \\ \rightarrowtail}\; \Delta^1 \;\substack{\rightarrowtail \\ \twoheadleftarrow \\ \rightarrowtail \\ \twoheadleftarrow \\ \rightarrowtail}\; \Delta^2 \;\substack{\rightarrowtail \\ \twoheadleftarrow \\ \rightarrowtail \\ \twoheadleftarrow \\ \rightarrowtail \\ \twoheadleftarrow \\ \rightarrowtail}\; \Delta^3 \qquad \cdots$$

each of the maps denoted by "$\rightarrowtail$" given by an ordered injection of the vertices and each of the maps denoted "$\twoheadrightarrow$" given by an ordered surjection of the vertices. By the Yoneda lemma, $\mathbb{\Delta}$ is isomorphic to the full subcategory of sSet spanned by the standard simplices; the functor $\Delta^\bullet \colon \mathbb{\Delta} \hookrightarrow \mathsf{sSet}$ sending $[n]$ to the standard n-simplex Δ^n is referred to as the *Yoneda embedding.*

The standard simplices and maps between them generate the category sSet under gluing. That is, any simplicial set X may be thought of as a triangulated space comprised of all of its simplices $\coprod_n X_n$ glued together along the face and degeneracy maps (I.2.2). For a more leisurely introduction to simplicial sets, see [R1].

A simplicial category is typically thought of as a category with objects together with mapping spaces (i.e., simplicial sets) between them. There is an alternate presentation of this data which will also be convenient in which an *n-simplex* in a mapping space from a to b is encoded as an *n-arrow* from a to b.

Definition I.2.3. A **simplicial category** $\mathsf{A}_\bullet$ is given by categories A_n for each $n \geq 0$ with a common set of objects $\mathrm{ob}\mathsf{A}$ and whose morphisms are called n-**arrows** that assemble into a diagram $\mathbb{\Delta}^{\mathrm{op}} \to \mathsf{Cat}$ of identity-on-objects functors.

$$\text{(I.2.4)} \qquad \mathsf{A}_\bullet := \qquad \mathsf{A}_0 \;\substack{\twoheadleftarrow \\ \rightarrowtail \\ \twoheadleftarrow}\; \mathsf{A}_1 \;\substack{\twoheadleftarrow \\ \rightarrowtail \\ \twoheadleftarrow \\ \rightarrowtail \\ \twoheadleftarrow}\; \mathsf{A}_2 \;\substack{\twoheadleftarrow \\ \rightarrowtail \\ \twoheadleftarrow \\ \rightarrowtail \\ \twoheadleftarrow \\ \rightarrowtail \\ \twoheadleftarrow}\; \mathsf{A}_3 \qquad \cdots$$

Proposition I.2.5. *The following are equivalent:*
- *a simplicial category* $\mathsf{A}_\bullet$ *with object set* $\mathrm{ob}\mathsf{A}$
- *a simplicially enriched category* A *with objects* $\mathrm{ob}\mathsf{A}$

Proof. For any $x, y \in \mathrm{ob}\mathsf{A}$, an n-arrow in $\mathsf{A}_n(x,y)$ corresponds to an n-simplex in the mapping space $\mathsf{A}(x,y)$. □

For any ordinary category A, we now introduce a simplicial category $\mathfrak{C}\mathsf{A}$ whose n-arrows parametrize the data of a homotopy coherent diagram of shape A.[4]

Definition I.2.6 (free resolutions)**.** Forgetting composition, let $U\mathsf{A}$ denote the underlying reflexive directed graph of a category A, and let $FU\mathsf{A}$ denote the free category on the underlying reflexive directed graph of A. It has the same objects as A and its non-identity arrows are strings of composable non-identity arrows of A.

[4]Cordier and Porter [CP1] write $\mathbf{S}(\mathsf{A})$ for the simplicial category $\mathfrak{C}\mathsf{A}$. Here we use notation that some readers might recognize from a related context. In Theorem II.2.4 to be proven in Part II, we prove that the object is isomorphic to this one.

We define a simplicial category $\mathfrak{C}\mathsf{A}_\bullet$ with $\mathrm{ob}\mathfrak{C}\mathsf{A} = \mathrm{ob}\mathsf{A}$ and with the category of n-arrows $\mathfrak{C}\mathsf{A}_n := (FU)^{n+1}\mathsf{A}$. A non-identity n-arrow is a string of composable arrows in A with each arrow in the string enclosed in exactly n pairs of well-formed parentheses. In the case $n = 0$, this recovers the previous description of the non-identity 0-arrows in $FU\mathsf{A}$, strings of composable non-identity arrows of A.

It remains to define the required identity-on-objects functors:[5]

$$\mathfrak{C}\mathsf{A}_\bullet := \qquad FU\mathsf{A} \;\substack{\twoheadleftarrow \\ \rightarrowtail \\ \twoheadleftarrow}\; (FU)^2\mathsf{A} \;\substack{\twoheadleftarrow \\ \rightarrowtail \\ \twoheadleftarrow \\ \rightarrowtail \\ \twoheadleftarrow}\; (FU)^3\mathsf{A} \;\substack{\twoheadleftarrow \\ \rightarrowtail \\ \twoheadleftarrow \\ \rightarrowtail \\ \twoheadleftarrow \\ \rightarrowtail \\ \twoheadleftarrow}\; (FU)^4\mathsf{A} \qquad \cdots$$

For $j \geq 1$, the face maps

$$(FU)^k\epsilon(FU)^j \colon (FU)^{k+j+1}\mathsf{A} \to (FU)^{k+j}\mathsf{A}$$

remove the parentheses that are contained in exactly k others, while $FU \cdots FU\epsilon$ composes the morphisms inside the innermost parentheses. For $j \geq 1$, the degeneracy maps

$$F(UF)^k\eta(UF)^jU \colon (FU)^{k+j+1}\mathsf{A} \to (FU)^{k+j+2}\mathsf{A}$$

double up the parentheses that are contained in exactly k others, while $F \cdots UF\eta U$ inserts parentheses around each individual morphism.

Example I.2.7. In the case of a discrete group G regarded as a one-object category $\mathsf{B}G$, the free resolution $\mathfrak{C}\mathsf{B}G$ is defined by specifying the single endo-hom-set of each category $(FU)^n\mathsf{B}G$, together with the composition action. The underlying graph of $\mathsf{B}G$ is given by the non-identity elements of G, and thus $(FU)\mathsf{B}G$ is the group of words in these letters, i.e., the free group on the non-identity elements of G. The group $(FU)^2\mathsf{B}G$ is then the group of words of words and so on.

Exercise I.2.8. Compute $\mathfrak{C}\omega$ and show that its n-arrows enumerate the data described above.[6]

The category A can also be thought of as a discrete simplicial category in which the diagram (I.2.4) is constant at A, so the only n-arrows are degenerated 0-arrows. There is a canonical "augmentation" map $\epsilon \colon \mathfrak{C}\mathsf{A} \to \mathsf{A}$ determined by its degree zero component $\epsilon \colon FU\mathsf{A} \to \mathsf{A}$ which is just given by composition in A.

Proposition I.2.9. *The functor $\epsilon \colon \mathfrak{C}\mathsf{A} \to \mathsf{A}$ is a local homotopy equivalence of simplicial sets. That is, for any pair of objects $x, y \in \mathsf{A}$, the map $\epsilon \colon \mathfrak{C}\mathsf{A}(x, y) \to \mathsf{A}(x, y)$ is a homotopy equivalence: $\mathfrak{C}\mathsf{A}(x, y)$ is homotopy equivalent to the discrete set $\mathsf{A}(x, y)$ of arrows in A from x to y.*

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[5] More concisely, the free and forgetful functors just described define an adjunction

$$\begin{tikzcd} \mathsf{rDirGph} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathsf{Cat} \arrow[l, bend left, "U"] \end{tikzcd}$$

between small categories and reflexive directed graphs inducing a comonad FU on Cat; see Definition III.4.1. The simplicial object $\mathfrak{C}\mathsf{A}_\bullet$ is defined by evaluating the comonad resolution for $(FU, \epsilon, F\eta U)$ on a small category A. The face and degeneracy maps are whiskerings of the unit and counit of the adjunction; hence the notation. This structure will reappear in Part III below.

[6] In fact, it has more n-arrows than the n-homotopies describe above. We will be able to explain this when we return to this example in Part II.

Proof. The augmented simplicial object

$$\begin{tikzcd}[column sep=large]
\mathsf{A} \arrow[r, dashed, bend right] & (FU)\mathsf{A} \arrow[l, two heads] \arrow[r, tail] \arrow[r, dashed, bend right, shift right=3] & (FU)^2\mathsf{A} \arrow[l, two heads, shift right=2] \arrow[l, two heads, shift left=2] \arrow[r, tail, shift left=1] \arrow[r, tail, shift right=1] \arrow[r, dashed, bend right, shift right=5] & (FU)^3\mathsf{A} \arrow[l, two heads] \arrow[l, two heads, shift right=3] \arrow[l, two heads, shift left=3] \arrow[r, tail] \arrow[r, tail, shift left=3] \arrow[r, tail, shift right=3] \arrow[r, dashed, bend right, shift right=7] & (FU)^4\mathsf{A} \arrow[l, two heads, shift left=2] \arrow[l, two heads, shift right=2] \arrow[l, two heads, shift left=5] \arrow[l, two heads, shift right=5] & \cdots
\end{tikzcd}$$

is split at the level of reflexive directed graphs (i.e., after applying U). These splittings are not functors, but that won't matter. These directed graph morphisms displayed here are all identity on objects, which means that for any x, y there is a split augmented simplicial set

$$\begin{tikzcd}
\mathsf{A}(x,y) \arrow[r, bend right] & (FU)\mathsf{A}(x,y) \arrow[l, two heads] \arrow[r, tail] \arrow[r, bend right, shift right=3] & (FU)^2\mathsf{A}(x,y) \arrow[l, two heads, shift right=2] \arrow[l, two heads, shift left=2] \arrow[r, tail, shift left=1] \arrow[r, tail, shift right=1] \arrow[r, bend right, shift right=5] & (FU)^3\mathsf{A}(x,y) \arrow[l, two heads] \arrow[l, two heads, shift right=3] \arrow[l, two heads, shift left=3] \arrow[r, tail] \arrow[r, tail, shift left=3] \arrow[r, tail, shift right=3] \arrow[r, bend right, shift right=7] & (FU)^4\mathsf{A}(x,y) \arrow[l, two heads, shift left=2] \arrow[l, two heads, shift right=2] \arrow[l, two heads, shift left=5] \arrow[l, two heads, shift right=5] \cdots
\end{tikzcd}$$

and now some classical simplicial homotopy theory proves the claim [M]. $\square$

I.3. Homotopy coherent diagrams and homotopy coherent natural transformations

Finally, we can give a precise definition of the key notion of a homotopy coherent diagram:

Definition I.3.1. A **homotopy coherent diagram** of shape A is a functor $\mathfrak{C}\mathsf{A} \to \mathbb{S}\mathsf{pace}$.

Example I.3.2. A strictly commutative diagram $F\colon \mathsf{A} \to \mathbb{S}\mathsf{pace}$ gives rise to a homotopy coherent diagram by composing with the augmentation map

$$\mathfrak{C}\mathsf{A} \xrightarrow{\epsilon} \mathsf{A} \xrightarrow{F} \mathbb{S}\mathsf{pace}. \tag{I.3.3}$$

In this case, all n-homotopies are identities.

Not every homotopy commutative diagram can be made homotopy coherent. The following counterexample was suggested by Thomas Kragh and communicated by Hiro Tanaka.

Example I.3.4. Let $p\colon E \to B$ be a Serre fibration with $i\colon F \to E$ the inclusion of the fiber over the basepoint $*$ of B. A diagram

$$\begin{tikzcd}
& X \arrow[dl, "f"'] \arrow[dr, "e"] \arrow[dd, "*"] & \\
F \arrow[rr, "i"] \arrow[dr, "*"'] & & E \arrow[dl, "p"] \\
& B &
\end{tikzcd}$$

is *homotopy commutative* if there exist homotopies $\alpha\colon e \simeq if$ and $\beta\colon pe \simeq *$, the other two triangles being strictly commutative. The diagram is then *homotopy coherent* if and only if there exists a 2-homotopy between $p\alpha\colon pe \simeq *$ and β. If this is the case, then since p is a Serre fibration it is possible to lift the 2-homotopy along p to define a homotopy $\alpha'\colon e \simeq if$ so that $p\alpha' = \beta$. Applying the universal property of the fiber F as the homotopy pullback of p along the inclusion of the basepoint, the homotopy β induces a map $g\colon X \to F$ and the homotopy α' then implies that f and g are homotopic.

Applying these observations in the case of the Hopf fibration, consider the diagram

$$\begin{tikzcd} & S^1 \arrow[dl, "n"'] \arrow[dr, "i"] \arrow[dd, "*"] & \\ S^1 \arrow[rr, "i"] \arrow[dr, "*"'] & & S^3 \arrow[dl, "p"] \\ & S^2 & \end{tikzcd}$$

involving a map $S^1 \to S^1$ of degree n. Since $\pi_1 S^3 = 0$, there exists a homotopy $\alpha\colon i \simeq in$. Both pi and pin equal the constant map $*$ at the basepoint of S^2, but $p\alpha$ is not 2-homotopic to the constant homotopy $*$, for if it were we would obtain a homotopy between the map of degree n and the identity map $S^1 \to S^1$, which does not exist for most n. Thus, this homotopy commutative diagram cannot be made homotopy coherent.

A natural transformation is a type of higher morphism between parallel functors. Natural transformations are analogous to homotopies with the category $[1] = 0 \to 1$ playing the role of the interval.

Definition I.3.5. Given a parallel pair of functors $F, G\colon \mathsf{C} \to \mathsf{D}$, a **natural transformation** $\alpha\colon F \to G$ is specified by a functor $\alpha\colon \mathsf{C} \times [1] \to \mathsf{D}$ that restricts on the "endpoints" of $[1]$ to F and G as follows:

$$\begin{tikzcd} \mathsf{C} \arrow[d, hook, "0"'] \arrow[dr, "F"] & \\ \mathsf{C} \times [1] \arrow[r, "\alpha"] & \mathsf{D} \\ \mathsf{C} \arrow[u, hook', "1"] \arrow[ur, "G"'] & \end{tikzcd}$$

This suggests the following definition of a homotopy coherent natural transformation.

Definition I.3.6. A **homotopy coherent natural transformation** $\alpha\colon F \to G$ between homotopy coherent diagrams F and G of shape A is a homotopy coherent diagram of shape $\mathsf{A} \times [1]$ that restricts on the endpoints of $[1]$ to F and G as follows:

$$\begin{tikzcd} \mathfrak{C}\mathsf{A} \arrow[d, hook, "0"'] \arrow[dr, "F"] & \\ \mathfrak{C}(\mathsf{A} \times [1]) \arrow[r, "\alpha"] & \mathbb{S}\mathsf{pace} \\ \mathfrak{C}\mathsf{A} \arrow[u, hook', "1"] \arrow[ur, "G"'] & \end{tikzcd}$$

Note that the data of a pair of homotopy coherent natural transformations $\alpha\colon F \to G$ and $\beta\colon G \to H$ between homotopy coherent diagrams of shape A does not uniquely determine a (vertical) "composite" homotopy coherent natural transformation $F \to H$ because this data does not define a homotopy coherent diagram of shape $\mathsf{A} \times [2]$, where $[2] = 0 \to 1 \to 2$.[7] This observation motivated Boardman and Vogt to define, in place of a *category* of homotopy coherent diagrams and natural transformations of shape A, a *quasi-category* of homotopy coherent diagrams and natural transformations of shape A.

Definition I.3.7. For any category A, let $\mathrm{Coh}(\mathsf{A}, \mathbb{S}\mathsf{pace})$ denote the simplicial set whose n-simplices are homotopy coherent diagrams of shape $\mathsf{A} \times [n]$, i.e., are simplicial functors

$$\mathfrak{C}(\mathsf{A} \times [n]) \to \mathbb{S}\mathsf{pace},$$

where $[n] \subset \omega$ denotes the category freely generated by the reflexive directed graph

$$[n] := \qquad 0 \longrightarrow 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots \longrightarrow n.$$

The simplicial category $\mathbb{S}\mathsf{pace}$ has an important property alluded to above: its mapping spaces $\mathrm{Map}(X, Y)$ are **Kan complexes**, simplicial sets in which any horn

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & \mathrm{Map}(X,Y) \\ \Delta^n \arrow[ur, dashed] \end{tikzcd}$$

with $0 \leq k \leq n$ may be filled to a simplex. Any simplicial category, such as $\mathbb{S}\mathsf{pace}$, extracted from a topologically-enriched category is automatically *Kan complex enriched* because its mapping spaces are defined as total singular complexes of topological spaces [R3, §16.1]. It is because of this property that:

Theorem I.3.8 ([BV])**.** *$\mathrm{Coh}(\mathsf{A}, \mathbb{S}\mathsf{pace})$ is a **quasi-category**, i.e., any inner horn*

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & \mathrm{Coh}(\mathsf{A}, \mathbb{S}\mathsf{pace}) \\ \Delta^n \arrow[ur, dashed] \end{tikzcd}$$

with $0 < k < n$ admits a filler.

Proof. This can be checked directly, or deduced as an immediate consequence — see Corollary II.2.8 — of a result that we will prove in Part II. ☐

This is the first example of a "weak Kan complex," the name Boardman and Vogt used those simplicial sets that admit fillers for inner horns, which Joyal rechristened "quasi-categories." Quasi-categories define a popular model of $(\infty, 1)$-*categories*, categories weakly enriched in topological spaces, about more which in Part III.

Any quasi-category has a **homotopy category** whose objects are the vertices and whose morphisms are 1-simplices up to a homotopy relation $f \simeq g$ between parallel 1-simplices $f, g \colon x \to y$ witnessed by a 2-simplex with boundary:

$$\begin{tikzcd} & y \arrow[dr, equal] & \\ x \arrow[ur, "f"] \arrow[rr, "g"'] & & y \end{tikzcd}$$

[7] In notation to be introduced in Part II, α and β define a diagram of shape $\mathfrak{C}(\mathsf{A} \times \Lambda^2_1)$ rather than a diagram of shape $\mathfrak{C}(\mathsf{A} \times [2])$, where Λ^2_1 is the shape of the generating reflexive directed graph of the category $[2]$.

Composition relations are also witnessed by 2-simplices: the homotopy class of $f\colon x \to y$ and the homotopy class of $g\colon y \to z$ compose to the homotopy class of $h\colon x \to z$ if and only if there is a 2-simplex whose boundary has the form

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & & z \end{tikzcd}$$

The following result was first proven by Vogt and then generalized by Cordier and Porter:

Theorem I.3.9 ([V, CP1])**.** *The natural map* $\mathbb{S}\mathsf{pace}^{\mathsf{A}} \to \mathrm{Coh}(\mathsf{A}, \mathbb{S}\mathsf{pace})$ *defined by* (I.3.3) *induces an equivalence of homotopy categories*

$$\mathsf{Ho}(\mathbb{S}\mathsf{pace}^{\mathsf{A}}) \xrightarrow{\simeq} \mathsf{Ho}\mathrm{Coh}(\mathsf{A}, \mathbb{S}\mathsf{pace}),$$

where $\mathsf{Ho}(\mathbb{S}\mathsf{pace}^{\mathsf{A}})$ *is defined by localizing at the componentwise homotopy equivalences.*

Part II. Homotopy coherent realization and the homotopy coherent nerve

Recall that a homotopy coherent diagram of shape A is a simplicial functor indexed by a category $\mathfrak{C}\mathsf{A}$ defined as a free resolution of A, a construction that will be reviewed momentarily. Explicitly, the data of such a diagram is comprised of objects X_a for each object $a \in \mathsf{A}$ plus maps of simplicial sets

$$\mathfrak{C}\mathsf{A}(a,b) \to \mathrm{Map}(X_a, X_b)$$

for each pair of objects that are functorial in the sense of commuting with the composition functions:

$$\begin{tikzcd} \mathfrak{C}\mathsf{A}(b,c) \times \mathfrak{C}\mathsf{A}(a,b) \arrow[r, "\circ"] \arrow[d] & \mathfrak{C}\mathsf{A}(a,c) \arrow[d] \\ \mathrm{Map}(X_b, X_c) \times \mathrm{Map}(X_a, X_b) \arrow[r, "\circ"] & \mathrm{Map}(X_a, X_c) \end{tikzcd}$$

Previously we interpreted X_a and X_b as spaces, but this interpretation is actually not necessarily. What we do need is for $\mathrm{Map}(X_a, X_b)$ to be a space, by which we mean a Kan complex, because it is in these mapping spaces that we are defining homotopies (as 1-simplices) and higher homotopies (as higher simplices). So henceforth, we will extend our notion of **homotopy coherent diagram** to encompass any simplicial functor $\mathfrak{C}\mathsf{A} \to \mathbb{S}$ whose codomain $\mathbb{S}$ is a category enriched in Kan complexes.[8] One choice is $\mathbb{S} = \mathbb{S}\mathsf{pace}$ but there are others. Note that any topological category can be made into a category enriched in Kan complexes so there are many examples.

Indeed, many (large) quasi-categories — e.g., of spaces, of spectra, or whatever — originate as categories enriched in Kan complexes by Theorem II.2.7 below. Our aim today is to explain how diagrams that are valued in quasi-categories that arise in this way are automatically homotopy coherent.

To justify this slogan, we offer a second perspective on the simplicial category $\mathfrak{C}\mathsf{A}$ defined as the free resolution of a category A, explaining its relationship to the

[8]Categories enriched in Kan complexes are called "locally Kan" in much of the literature.

famous *homotopy coherent nerve* functor. This work will also allow us to generalize the indexing shapes for homotopy coherent diagrams to encompass simplicial sets which may or may not be nerves of categories. This will allow us to distinguish, e.g., between the ordinal category [2] and its generating reflexive directed graph Λ^2_1.

II.1. Free resolutions are simplicial computads

An arrow in a category is **atomic** if it is not an identity and if it admits no non-trivial factorizations, i.e., if whenever $f = g \circ h$ then one or other of g and h is an identity. A category is **freely generated** by a reflexive directed graph of atomic arrows if and only if each of its non-identity arrows may be uniquely expressed as a composite of atomic arrows.[9]

The following definition is due to Verity [RV6].[10]

Definition II.1.1 (simplicial computad)**.** A simplicial category A is a *simplicial computad* if and only if:

- each category A_n of n-arrows is freely generated by the graph of atomic n-arrows
- if f is an atomic n-arrow in A_n and $\alpha\colon [m] \twoheadrightarrow [n]$ is an epimorphism in $\mathbb{\Delta}$ then the degenerated m-arrow $f \cdot \alpha$ is atomic in A_m.

Lemma II.1.2. *A simplicial category* A *is a simplicial computad if and only if all of its non-identity arrows* f *can be expressed uniquely as a composite*

$$f = (f_1 \cdot \alpha_1) \circ (f_2 \cdot \alpha_2) \circ \cdots \circ (f_\ell \cdot \alpha_\ell)$$

in which each f_i *is non-degenerate and atomic and each* $\alpha_i \in \mathbb{\Delta}$ *is a degeneracy operator.*

Proof. This characterization follows immediately from the definition by applying the Eilenberg-Zilber lemma [GZ, II.3.1, pp. 26-27], which says that any degenerate simplex in a simplicial set may be uniquely expressed as a degenerated image of a non-degenerate simplex. □

Free resolutions define simplicial computads, whose atomic n-arrows index the generating n-homotopies in a homotopy coherent diagram, such as enumerated for the homotopy coherent simplex in §I.2.

Proposition II.1.3. *The free resolution* $\mathfrak{C}\mathsf{A}$ *is a simplicial computad.*

Proof. Recall $\mathfrak{C}\mathsf{A}$ is defined to be the free resolution of A, whose category of n-arrows is $(FU)^{n+1}\mathsf{A}$. The category $FU\mathsf{A}$ is the free category on the underlying reflexive directed graph of A. Its arrows are strings of composable non-identity arrows of A; the atomic 0-arrows are the non-identity arrows of A. An n-arrow is a string of composable arrows in A with each arrow in the string enclosed in exactly n pairs of parentheses. The atomic n-arrows are those enclosed in precisely one pair of parentheses on the outside. Since composition in a free category is by concatenation, the unique factorization property is clear. Since degeneracy arrows "double up" on parentheses, these preserve atomics as required. □

[9] This is the case just when the category is in the essential image of the free category functor $F\colon \mathsf{rDirGph} \to \mathsf{Cat}$.

[10] The reader familiar with model categorical intuition might find it helpful to note that the simplicial computads are precisely the cofibrant objects in the Bergner model structure on simplicially enriched categories; see [R3, §16.2] for proof.

We will now do the homework assigned in Part I.

Example II.1.4. Recall the category

$$\omega := \qquad 0 \longrightarrow 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots$$

The free resolution $\mathfrak{C}\omega$ has objects $n \geq 0$.

- A 0-arrow from j to k is a sequence of non-identity composable morphisms from j to k, the data of which is uniquely determined by the objects being passed through. So 0-arrows from j to k correspond to subsets
$$\{j,k\} \subset T^0 \subset [j,k]$$
of the closed internal $[j,k] = \{t \in \omega \mid j \leq t \leq k\}$ containing both endpoints.
- A 1-arrow from j to k is a once bracketed sequence of non-identity composable morphisms from j to k. This data is specified by two nested subsets
$$\{j,k\} \subset T^1 \subset T^0 \subset [j,k]$$
the larger one T^0 specifying the underlying unbracketed sequence and the smaller one T^1 specifying the placement of the brackets.[11]
- A n-arrow from j to k is an n times bracketed sequence of non-identity composable morphisms from j to k, the data of which is specified by nested subsets
$$\{j,k\} \subset T^n \subset \cdots \subset T^0 \subset [j,k]$$
indicating the locations of all of the parentheses.[12]

What then are the mapping spaces $\mathfrak{C}\omega(j,k)$? When $j > k$ they are empty and when $k = j$ or $k = j+1$ we have $\{j,k\} = [j,k]$ so $\mathfrak{C}\omega(j,k) \cong \Delta^0$ is comprised of a single point. For $k > j$, there are $k-j-1$ elements of $[j,k]$ excluding the endpoints and so we see that $\mathfrak{C}\omega(j,k)$ has 2^{k-j-1} vertices. The n-simplices of $\mathfrak{C}\omega(j,k)$ are given by specifying $n+1$ vertices — each a subset $\{j,k\} \subset T^i \subset [j,k]$ — that respect the ordering of subsets relation. From this we see that

$$\mathfrak{C}\omega(j,k) \cong (\Delta^1)^{k-j-1},$$ [13]

as displayed for instance in the case $j = 0$ and $k = 4$:

$$\mathfrak{C}\omega(0,4) := \left(\begin{tikzcd}
& \{0,4\} \arrow[rr] \arrow[dl] \arrow[dd] \arrow[dddl, dotted] \arrow[dr, dotted] \arrow[ddrr, dotted] \arrow[dddr, dotted] & & \{0,1,4\} \arrow[dl] \arrow[dd] \arrow[dddl, dotted] \\
\{0,3,4\} \arrow[rr] \arrow[dd] \arrow[ddrr, dotted] & & \{0,1,3,4\} \arrow[dd] & \\
& \{0,2,4\} \arrow[rr] \arrow[dl] \arrow[dr, dotted] & & \{0,1,2,4\} \arrow[dl] \\
\{0,2,3,4\} \arrow[rr] & & \{0,1,2,3,4\} &
\end{tikzcd}\right)$$

---

[11]Note the face and degeneracy maps $(\mathfrak{C}\omega)_0 \mathrel{\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow}} (\mathfrak{C}\omega)_1$ are the obvious ones, either duplicating or omitting one of the sets T^i.

[12]The nesting is because parenthezations should be "well formed" with open brackets closed in the reverse order to that in which they were opened.

[13]More explicitly, this argument shows that the simplicial set $\mathfrak{C}\omega(j,k)$ is the nerve of the poset of subsets $\{j,k\} \subset T \subset [j,k]$ ordered by inclusion.

The simplicial category $\mathfrak{C}\omega$ is a simplicial computad whose atomic n-arrows are those with a single outermost parenthezation: i.e., for which $T^n = \{j,k\}$. Geometrically these are all the simplices in the hom space cube $(\Delta^1)^{k-j-1}$ that contain the initial vertex $\{j,k\}$.

II.2. Homotopy coherent realization and the homotopy coherent nerve

Employing topological notation, we write $[n] \subset \omega$ for the full subcategory spanned by $0,\ldots,n$.

$$[n] := \qquad 0 \longrightarrow 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots \longrightarrow n$$

These categories define the objects of a diagram $\mathbb{\Delta} \hookrightarrow \mathsf{Cat}$ that is a full embedding: the only functors $[m] \to [n]$ are order-preserving maps from $[m] = \{0,\ldots,m\}$ to $[n] = \{0,\ldots,n\}$. Applying the free resolution construction to these categories we get a functor $\mathfrak{C}\colon \mathbb{\Delta} \to \mathsf{sCat}$ where $\mathfrak{C}[n]$ is the full simplicial subcategory of $\mathfrak{C}\omega$ spanned by those objects $0,\ldots,n$. In particular, its hom spaces are the simplicial cubes described in Example II.1.4.

Definition II.2.1 (homotopy coherent realization and nerve)**.** The homotopy coherent nerve $\mathfrak{N}$ and homotopy coherent realization $\mathfrak{C}$ are the adjoint pair of functors obtained by applying Kan's construction [R3, 1.5.1] to the functor $\mathfrak{C}\colon \mathbb{\Delta} \to \mathsf{sCat}$ to construct an adjunction

$$\begin{tikzcd} \mathsf{sSet} \arrow[r, bend left, "\mathfrak{C}"] \arrow[r, phantom, "\perp"] & \mathsf{sCat} \arrow[l, bend left, "\mathfrak{N}"] \end{tikzcd}$$

The right adjoint, called the **homotopy coherent nerve**, is defined at a simplicial category $\mathbb{S}$ by defining the n-simplices of the simplicial set $\mathfrak{N}\mathbb{S}$ to be homotopy coherent diagrams of shape $[n]$ in $\mathbb{S}$. That is

$$\mathfrak{N}\mathbb{S}_n := \{\mathfrak{C}[n] \to \mathbb{S}\}.$$

The left adjoint is defined by pointwise left Kan extension along the Yoneda embedding of I.2.1:

$$\begin{tikzcd} \mathbb{\Delta} \arrow[rr, hook, "\Delta^\bullet"] \arrow[dr, "\mathfrak{C}"'] & \cong & \mathsf{sSet} \arrow[dl, dashed, "\mathfrak{C}"] \\ & \mathsf{sCat} & \end{tikzcd}$$

That is, $\mathfrak{C}\Delta^n$ is defined to be $\mathfrak{C}[n]$ — a simplicial category that we call the **homotopy coherent n-simplex** — and for a generic simplicial set X, $\mathfrak{C}X$ is defined to be a colimit of the homotopy coherent simplices indexed by the category of simplices of X.[14]

Because of the formal similarity with the geometric realization functor, another left adjoint defined by Kan's construction, we refer to $\mathfrak{C}$ as **homotopy coherent realization**.

[14]The simplicial set X is obtained by gluing in a Δ^n for each n-simplex $\Delta^n \to X$ of X. The functor $\mathfrak{C}$ preserves these colimits, so $\mathfrak{C}X$ is obtained by gluing in a $\mathfrak{C}[n]$ for each n-simplex of X.

Left Kan extensions are computed as colimits, providing a formula for the homotopy coherent realization $\mathfrak{C}X$ of a simplicial set X as the colimit of a diagram of homotopy coherent simplices $\mathfrak{C}[n]$. However, this does not give very much insight into the mapping space of $\mathfrak{C}X$, colimits of simplicial categories being rather complicated. Work of Dugger and Spivak [DuSp], redeveloped in [RV6, §4], fills this gap; see also [R3, §16.2-16.4].

Proposition II.2.2 ([RV6, 4.4.7])**.** *For any simplicial set X, its homotopy coherent realization $\mathfrak{C}X$ is a simplicial computad in which:*

- *objects* $\mathrm{ob}\mathfrak{C}X = X_0$*, the vertices of the simplicial set X*
- *atomic* 0*-arrows are non-degenerate 1-simplices of X, the source being the initial vertex and the target being the final vertex of the simplex*
- *atomic* 1*-arrows are the non-degenerate k-simplices of X for $k > 1$, the source being the initial vertex and the target being the final vertex of the simplex*
- *atomic n-arrows are pairs comprised of a non-degenerate k-simplex in X for some $k > n$ together with a set of proper inclusions*

$$\{0, k\} = T^n \subsetneq T^{n-1} \cdots \subsetneq T^0 = [0, k] \tag{II.2.3}$$

the data of which defines an atomic n-arrow in $\mathfrak{C}\Delta^k$ from 0 *to k that is not in the image of any of the face maps. This source of this n-arrow is the initial vertex of the k-simplex, while the target is the final vertex of the simplex.*

Note that the description of atomic n-arrows subsumes those of the atomic 0-arrows and atomic 1-arrows. The data of a non-degenerate atomic n-arrow from x to y in $\mathfrak{C}X$ is given by a "bead," that is a non-degenerate k-simplex in X from x to y, together with the additional data of a sequence of proper subset inclusions (II.2.3), which Dugger and Spivak refer to as a "flag of vertex data." Non-atomic n-arrows are then "necklaces," that is strings of beads in X joined head to tail, together with accompanying "vertex data" for each simplex.

Proof sketch. This can be proven inductively using the skeletal decomposition of the simplicial set X and will reveal that for any inclusion of simplicial sets $X \subset Y$, the functor $\mathfrak{C}X \hookrightarrow \mathfrak{C}Y$ of homotopy coherent realizations is a **simplicial subcomputad inclusion**: a functor of simplicial computads that is injective on objects and faithful and also preserves atomic arrows. This is proven by verifying directly that $\mathfrak{C}\partial\Delta^k \hookrightarrow \mathfrak{C}\Delta^k$ is a simplicial subcomputad inclusion and then arguing that such inclusions are closed under coproduct, pushout, and transfinite composition in simplicial categories. It follows that $\mathfrak{C}X$ is a simplicial computad, and moreover the analysis of what happens when attaching an k-simplex provides the description of atomic n-arrows given above. The data (II.2.3) represents an n-arrows in $\mathfrak{C}\Delta^k(0, k)$ that is atomic (since $T^n = \{0, k\}$) and not contained in any face (since $T^0 = [0, k]$). $\square$

Applying Kan's construction to the embedding $\mathbb{\Delta} \hookrightarrow \mathsf{Cat}$ of the ordinal categories yields an adjunction

$$\begin{tikzcd} \mathsf{sSet} \arrow[r, bend left, "\mathsf{Ho}"] \arrow[r, phantom, "\perp"] & \mathsf{Cat} \arrow[l, bend left] \end{tikzcd}$$

the right adjoint of which is called the **nerve** and the left adjoint of which, defined by pointwise left Kan extension along the Yoneda embedding:

$$\begin{tikzcd} \mathbb{\Delta} \arrow[rr, hook, "\Delta^\bullet"] \arrow[dr] & \arrow[d, phantom, "\cong"] & \mathsf{sSet} \arrow[dl, dashed, "\mathsf{Ho}"] \\ & \mathsf{Cat} & \end{tikzcd}$$

defines the homotopy category of a simplicial set, via a mild generalization of the construction introduced for quasi-categories at the end of Part I. For a category A, an n-simplex in the nerve of A is simply a functor $[n] \to \mathsf{A}$, i.e., a string of n-composable morphisms in A. Note that by fullness of $\mathbb{\Delta} \hookrightarrow \mathsf{Cat}$, the nerve of the ordinal category $[n]$ is the standard n-simplex Δ^n.

The nerve functor defines a fully faithful embedding $\mathsf{Cat} \hookrightarrow \mathsf{sSet}$ of categories into simplicial sets that lands in the subcategory spanned by the quasi-categories. In quasi-category theory, it is very convenient to conflate a category with its nerve, which is why we have not introduced notation for this right adjoint.[15]

With this convention, we now have two simplicial categories we have denoted $\mathfrak{C}\mathsf{A}$ for a small category A: the free resolution of A and the homotopy coherent realization of the nerve of A. This would be confusing were these objects not naturally isomorphic:[16]

Theorem II.2.4 ([R2, 6.7])**.** *For any category* A*,* $\mathfrak{C}\mathsf{A} \cong \mathfrak{C}\mathsf{A}$*. That is, its free resolution is naturally isomorphic to the homotopy coherent realization of its nerve.*

Remark II.2.5. Note $\mathfrak{C}\Delta^n \cong \mathfrak{C}[n]$ is tautologous. The left Kan extension along the Yoneda embedding is defined so as to agree with $\mathfrak{C}\colon \mathbb{\Delta} \to \mathsf{sCat}$ on the subcategory of representables. Many arguments involving simplicial sets can be reduced to a check on representables, with the extension to the general case following formally by "taking colimits." This result, however, is not one of them since we are trying to prove something for all *categories* and the embedding $\mathsf{Cat} \hookrightarrow \mathsf{sSet}$ does not preserve colimits.

Proof. Proposition II.1.3 and Proposition II.2.2 reveal that both simplicial categories are simplicial computads. We will argue that they have the same objects and non-degenerate atomic n-arrows.

Both have $\mathrm{ob}\mathsf{A}$ as objects, these being the vertices in the nerve of A. Atomic 0-arrows of the free resolution are morphisms in A; while atomic 0-arrows in the coherent realization are non-degenerate 1-simplices of the nerve — these are the same thing. Atomic non-degenerate 1-arrows of the free resolution are sequences of at least two morphisms (enclosed in a single set of outer parentheses), while atomic 1-arrows of the coherent realization are non-degenerate simplices of dimension at least two — again these are the same. Finally a non-degenerate atomic n-arrow is a sequence of k composable morphisms with $(n-1)$ non-repeating bracketings; this non-degeneracy necessitates $k > n$. This data defines a k-simplex in the nerve together with a non-degenerate atomic n-arrow in $\mathfrak{C}[k](0,k)$, i.e., an atomic n-arrow in the coherent realization. □

[15]Note that the nerve of the category $\mathsf{B}G$ with a single object and elements of the group g as its endomorphisms is the Kan complex that typically goes by this name.

[16]Corollary II.2.6, which implies Theorem II.2.4, is stated without proof in [CP1]. The argument given here appears in [R2], though it is highly probable that an earlier proof exists in the literature.

Because the homotopy coherent realization was defined to be the left adjoint to the homotopy coherent nerve, it follows immediately:

Corollary II.2.6.

(i) *Homotopy coherent diagrams of shape* A *in a simplicial category* $\mathbb{S}$ *correspond to maps from the nerve of* A *to the homotopy coherent nerve of* $\mathbb{S}$*: i.e., there is a natural bijection between simplicial functors* $\mathfrak{C}\mathsf{A} \to \mathbb{S}$ *and simplicial maps* $\mathsf{A} \to \mathfrak{N}\mathbb{S}$*.*

(ii) *Hence, the simplicial set* $\mathrm{Coh}(\mathsf{A}, \mathbb{S})$ *is isomorphic to the simplicial set* $\mathfrak{N}\mathbb{S}^{\mathsf{A}}$ *defined using the internal hom in* $\mathsf{s}\mathbb{S}\mathsf{et}$*.*

Note that the homotopy coherent realization functor is defined on all simplicial sets X. Extending previous terminology, we refer to a simplicial functor $\mathfrak{C}X \to \mathbb{S}$ as a **homotopy coherent diagram of shape** X **in** $\mathbb{S}$.

The simplicial computad structure of Proposition II.2.2 can also be used to prove the following important result:

Theorem II.2.7 ([CP1, 2.1])**.** *If* $\mathbb{S}$ *is Kan complex enriched, then* $\mathfrak{N}\mathbb{S}$ *is a quasi-category.*

Proof. By adjunction, to extend along an inner horn inclusion $\Lambda^n_k \hookrightarrow \Delta^n$ mapping into the homotopy coherent nerve $\mathfrak{N}\mathbb{S}$ is to extend along simplicial subcomputad inclusions $\mathfrak{C}\Lambda^n_k \hookrightarrow \mathfrak{C}\Delta^n$ mapping into the Kan complex enriched category $\mathbb{S}$. This is the simplicial subcomputad generated by all arrows whose beads are supported by simplices in $\Lambda^n_k \subset \Delta^n$. The only missing ones are in the mapping space from 0 to n, so we are asked to solve a single lifting problem

$$\begin{tikzcd} \mathfrak{C}\Lambda^n_k(0,n) \arrow[r] \arrow[d, hook] & \mathrm{Map}(X_0, X_n) \\ \mathfrak{C}\Delta^n(0,n) \arrow[ur, dashed] & \end{tikzcd}$$

In Example II.1.4, we have seen that $\mathfrak{C}\Delta^n(0,n) \cong (\Delta^1)^{n-1}$ is a cube. One can similarly check that $\mathfrak{C}\Lambda^n_k(0,n)$ is a cubical horn. Cubical horn inclusions can be filled in the Kan complex $\mathrm{Map}(X_0, X_n)$, completing the proof. □

Corollary II.2.8. $\mathrm{Coh}(\mathsf{A}, \mathbb{S}) \cong \mathfrak{N}\mathbb{S}^{\mathsf{A}}$ *is a quasi-category.*

Proof. By the adjunction of Definition II.2.1, a simplicial functor $\mathfrak{C}\mathsf{A} \to \mathbb{S}$ is the same as a simplicial map $\mathsf{A} \to \mathfrak{N}\mathbb{S}$. So $\mathrm{Coh}(\mathsf{A}, \mathbb{S}) \cong \mathfrak{N}\mathbb{S}^{\mathsf{A}}$ and since the quasi-categories define an exponential ideal in simplicial sets, the fact that $\mathfrak{N}\mathbb{S}$ is a quasi-category implies that $\mathfrak{N}\mathbb{S}^{\mathsf{A}}$ is too. □

Remark II.2.9 (all diagrams in homotopy coherent nerves are homotopy coherent). This corollary explains that any map of simplicial sets $X \to \mathfrak{N}\mathbb{S}$ transposes to define a simplicial functor $\mathfrak{C}X \to \mathbb{S}$, a homotopy coherent diagram of shape X in $\mathbb{S}$. While not every quasi-category is isomorphic to a homotopy coherent nerve of a Kan complex enriched category, a hard theorem shows that every quasi-category is equivalent to a homotopy coherent nerve; one proof appears as [RV6, 7.2.2]. This explains the slogan introduced at the beginning of this lecture, that all diagrams in quasi-categories are homotopy coherent.

II.3. Further applications

Using similar techniques, where a homotopy coherent realization problem is transposed along the adjunction $\mathfrak{C} \dashv \mathfrak{N}$ to an extension problem in simplicial sets mapping into the homotopy coherent nerve, one can show:

Proposition II.3.1 ([CP2, §2])**.** *Given a homotopy coherent diagram $F \colon \mathfrak{C}\mathsf{A} \to \mathbb{S}$ in a locally Kan simplicial category and a family of homotopy equivalences $f_a \colon Fa \to Ga$ for all $a \in \mathsf{A}$, there is a homotopy coherent diagram G and coherent map $f \colon F \to G$ that moreover defines an isomorphism in $\mathsf{Coh}(\mathsf{A}, \mathbb{S})$.*

Proof. By colimits reduce to the case $\mathsf{A} = \Delta^n$ and construct the desired extension

$$\begin{tikzcd} \Delta^n \times \Lambda^1_1 \cup \mathrm{sk}_0\Delta^n \times \Delta^1 \arrow[r] \arrow[d, hook] & \mathfrak{N}\mathbb{S} \\ \Delta^n \times \Delta^1 \arrow[ur, dashed] & \end{tikzcd}$$

See [CP2, §2] for a very explicit description of what this filling process looks like in low dimensions. □

Proposition II.3.2 ([CP2, §3])**.** *Given a homotopy coherent map $f \colon F \to G$ of homotopy coherent diagrams $F, G \colon \mathfrak{C}\mathsf{A} \to \mathbb{S}$ in a locally Kan simplicial category and a family of homotopies $f_a \simeq g_a \colon Fa \to Ga$ for all $a \in \mathsf{A}$, there is a homotopy coherent map $g \colon F \to G$ extending these component maps together with a coherent homotopy of homotopy coherent maps $H \colon \mathfrak{C}(\mathsf{A} \times [2]) \to \mathbb{S}$.*

Proof. The argument is analogous to the previous inductive one solving the lifting problems:

$$\begin{tikzcd} \Delta^n \times \Lambda^2_0 \cup \mathrm{sk}_0\Delta^n \times \Delta^2 \arrow[r] \arrow[d, hook] & \mathfrak{N}\mathbb{S} \\ \Delta^n \times \Delta^2 \arrow[ur, dashed] & \end{tikzcd}$$

□

Part III. Homotopy coherent algebra

In §II.2, we learned that if A is a category, then its nerve defines the shape of a homotopy coherent diagram taking values in a quasi-category. Today we will extend this principle, to argue that if $\mathbb{A}$ is a 2-category, then its "local nerve" (taking the nerve of the hom-categories to produce a simplicial category) defines the shape of homotopy coherent categorical structure in a quasi-categorically enriched category — at least in the case where the local nerve of $\mathbb{A}$ happens to be a simplicial computad. We will pursue this line of thought in two particular examples, where $\mathbb{A}$ indexes a *monad* or an *adjunction*.

III.1. From coherent homotopy theory to coherent category theory

We have argued that any category of suitably defined spaces is enriched over *Kan complexes*, simplicial sets in which any horn can be filled to a simplex. Writing $\mathrm{Map}(X, Y)$ for the mapping objects in such a category $\mathbb{S}$, we interpret the 0-arrows as functions $X \to Y$. The Kan complex property implies that all n-arrows for $n > 0$ are "invertible" in a suitable sense, so we interpret an n-arrow in a mapping

Kan complex $\mathrm{Map}(X, Y)$ as an n-homotopy, with the case $n = 1$ defining ordinary homotopies between parallel functions $f, g \colon X \to Y$. The totality of the data of $\mathbb{S}$ may be thought of as defining an $(\infty, 1)$-*category*, meaning a category with objects and morphisms in each dimension, all but the lowest of which are invertible.

There is another context in which we are used to having multiple dimensions of morphisms, namely category theory itself. Famously, the category Cat of ordinary categories and functors is a *2-category*. Here the morphisms between morphisms are the natural transformations of Definition I.3.5. As mentioned there, natural transformations are analogous to homotopies in the sense that they can be expressed as functors $H \colon \mathsf{C} \times [1] \to \mathsf{D}$ defined using the interval category $[1] = 0 \to 1$, but unlike homotopies natural transformations are not typically invertible, an important amount of extra flexibility.

Thus, the appropriate context for homotopy coherent category theory will be a category that is simplicially enriched but with two non-invertible dimensions of morphisms rather than just one. More precisely, a categorical context for homotopy coherent category theory is a simplicial category $\mathbb{K}$ that is *quasi-categorically enriched* as opposed to Kan complex enriched, in which case it is traditional to write $\mathrm{Fun}(X, Y)$ for the function complexes instead of $\mathrm{Map}(X, Y)$.

Example III.1.1. The categories of quasi-categories, Segal categories, complete Segal spaces, and naturally marked simplicial sets (1-complicial sets) are all enriched over quasi-categories.

Example III.1.2. Suitably defined categories of fibrations (isofibrations, cartesian fibrations, cocartesian fibrations) of any of these over a fixed base are also enriched over quasi-categories.

In each category just mentioned, the objects are a model of an $(\infty, 1)$-category — which, in deference to Lurie, most people call ∞-*categories* — or a fibered variant of the above. So if we develop homotopy coherent category theory in the context of any category enriched over quasi-categories, we are doing "model independent ∞-category" in a rather strict framework.

Digression III.1.3 (on model independent ∞-category theory). Some people use "model independent ∞-category theory" to refer either to some sort of hand-wavy ∞-category theory or to something that is secretly quasi-category theory but presented in somewhat different language. The idea in both cases is that an expert could make everything precise. As an aesthetic and expository philosophy, this approach makes a lot of sense, but my concern at present is that there may be too few "experts."[17]

While the field remains in its infancy, I prefer a more conservative deployment of model independent ∞-category theory that refers to mathematics that can be:

- specialized to the case of quasi-categories and so recover a theory that is compatible with the theory of Joyal and Lurie and
- specialized to other models of $(\infty, 1)$-categories and recover something equivalent to this in the sense that it is preserved and reflected by "change of universe functors."[18]

[17]A quick test to gauge the level of expertise of an interlocutor is to ask whether they can construct a model of the Yoneda embedding.

[18]The right Quillen equivalences between quasi-categories, Segal categories, complete Segal spaces, and 1-complicial sets established by Joyal and Tierney [JT] all define *biequivalences* of

This is the sense in which "model independent ∞-category" will be used here, referring to constructions and theorems that can be used for a variety of models of $(\infty, 1)$-categories in exactly the same way in each instance and without relying upon any details of the models, except to know that each category of ∞-categories is enriched of quasi-categories.[19]

A final example of a quasi-categorically enriched category is worth mentioning:

Example III.1.4. The category $\mathbb{C}\mathsf{at}$ of categories is self-enriched: for any pair of small category C and D, we may define the category D^{C} of functors and natural transformations. Passing to the nerve, this defines a quasi-categorically enriched category of categories, since nerves of ordinary categories are quasi-categories.

III.2. Monads in category theory

A **monad** on a category B is a syntactic way of encoding "algebraic structure" that might be borne by objects in B. Various other mechanisms for describing finitary algebraic operations satisfying equations exist — for instance operads or Lawvere theories — but monads are able to capture more general varieties of algebraic structure.

Definition III.2.1. A **monad** on B is given by an endofunctor $T\colon \mathsf{B} \to \mathsf{B}$ together with a pair of natural transformations $\eta\colon \mathrm{id}_{\mathsf{B}} \to T$ and $\mu\colon T^2 \to T$ so that the following "associativity" and "unit" diagrams commute.

$$\begin{tikzcd} T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"'] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"'] & T \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, "T\eta"] \arrow[dr, equal] & T^2 \arrow[d, "\mu"] & T \arrow[l, "\eta T"'] \arrow[dl, equal] \\ & T & \end{tikzcd}$$

Example III.2.2. Let $\{P_n\}_{n\in\mathbb{N}}$ be a symmetric operad in sets. Then if B has finite products and the colimits displayed below, the associated monad is defined for $b \in \mathsf{B}$ by

$$Tb := \sum_{n\in\mathbb{N}} P_n \times_{\Sigma_n} b^n.$$

We will review the definition of an **algebra** for a monad T on B later on. For now, suffice it to mention that the algebras for the monad construction in Example III.2.2 are equivalent to the algebras for the operad.

There are also more exotic monads. To mention just one example, there is a monad on the large quasi-category $\prod_{\mathrm{ob}B} \mathbb{Q}\mathsf{cat}$ whose algebras are those $\mathrm{ob}B$-indexed families of quasi-categories that assemble into the fibers for a cartesian fibration over the quasi-category B [RV7].

III.3. Homotopy coherent monads

A monad, as just defined, is a diagram inside $\mathbb{C}\mathsf{at}$ whose image is comprised of

- a single object, the category B on which the monad acts,
- a 0-arrow T, the monad endofunctor,

---

quasi-categorically enriched categories: functors that are surjective on objects up to equivalence and define a local equivalence of quasi-categories.

[19] For a considerably more detailed account of what it means to develop ∞-category theory "model independently" from this point of view see [RV].

- a pair of 1-arrows $\eta\colon \mathrm{id}_B \to T$ and $\mu\colon T^2 \to T$, the monad natural transformations,

satisfying the axioms of Definition III.2.1.

Let us try to naively conjure the data of a *homotopy coherent monad* before stating the full definition. That is, let us try to define a simplicial computad $\mathbb{M}\mathsf{nd}$ so that simplicial functors $\mathbb{M}\mathsf{nd} \to \mathbb{K}$ valued in a quasi-categorically enriched category define a monad on an object in $\mathbb{K}$.

Firstly, $\mathbb{M}\mathsf{nd}$ should have a single object, which we denote by $+$, whose image identifies the object of $\mathbb{K}$ on which the monad acts. Since $\mathbb{M}\mathsf{nd}$ has a single object we only need to describe the simplicial set $\mathbb{M}\mathsf{nd}(+,+)$ of endo-arrows.

There is a single generating 0-arrow t, whose image defines the endofunctor of the monad. Then the 0-arrows are t^n for all $n \geq 0$ with $t^0 = \mathrm{id}_+$.

Among the generating 1-arrows we should have $\eta\colon \mathrm{id}_+ \to t$ and $\mu\colon t^2 \to t$. But our intuition is that to define a "homotopy coherent" algebraic structure, we should avoid making unnecessary choices. This suggests that it would be better to have a generating 1-arrow $\mu_n\colon t^n \to t$ for all $n \geq 0$, the n-ary multiplication map, where the case $\mu_1 = \mathrm{id}_t$ and μ_0 we think of as the unit η.

By "horizontally" composing the atomic 1-arrows $\mu_{i_1}, \ldots, \mu_{i_m}$ we obtain composite 1-arrows $t^{\sum_j i_j} \to t^m$ defined as follows:

$$\begin{tikzcd} + \arrow[r, bend left, "t^{i_1}"] \arrow[r, bend right, "t"'] \arrow[r, phantom, "\Downarrow\mu_{i_1}"] & + \arrow[r, bend left, "t^{i_2}"] \arrow[r, bend right, "t"'] \arrow[r, phantom, "\Downarrow\mu_{i_2}"] & + & \cdots & + \arrow[r, bend left, "t^{i_m}"] \arrow[r, bend right, "t"'] \arrow[r, phantom, "\Downarrow\mu_{i_m}"] & + \end{tikzcd}$$

Because each of the generating 1-arrows μ_n has codomain t, these composite 1-arrows are uniquely determined by interpreting the codomain t^m as m copies of t each of which receives some map μ_i. The data of the map $\mu[\alpha]\colon t^n \to t^m$ is then given by an order-preserving map $\alpha\colon \{0, \ldots, n-1\} \to \{0, \ldots, m-1\}$, which can be thought of as a specification of the cardinality of the fiber over each $j \in \{0, \ldots, m-1\}$. This gives a complete description of the 1-arrows in $\mathbb{M}\mathsf{nd}(+,+)$.

What 2-arrows should there be in $\mathbb{M}\mathsf{nd}(+,+)$? Associativity says that "all maps from t^n to t should agree.". In a homotopy coherent context, relations become data witnessed by arrows of the next dimension up. This suggests that for any $n, m \geq 0$ and any simplicial map $\alpha\colon \{0, \ldots, n-1\} \to \{0, \ldots, m-1\}$ we should have a 2-arrow with boundary

$$\begin{tikzcd} & t^m \arrow[dr, "\mu_m"] & \\ t^n \arrow[ur, "\mu[\alpha]"] \arrow[rr, "\mu_n"'] & & t \end{tikzcd}$$

None of these relations implies the other so they should all be generating. The composite 2-arrows are then of the form

$$\begin{tikzcd} & t^m \arrow[dr, "\mu[\beta]"] & \\ t^n \arrow[ur, "\mu[\alpha]"] \arrow[rr, "\mu[\gamma]"'] & & t^k \end{tikzcd}$$

for $k > 1$ whenever γ and $\beta\alpha$ define the same function $[n-1] \to [k-1]$. A similar description can be given for the m-arrows for $m \geq 3$.

So what is $\mathbb{M}\mathsf{nd}(+,+)$? It has 0-arrows indexed by natural numbers $n \geq 0$, 1-arrows corresponding to all order preserving functions, 2-arrows corresponding to composable pairs of order preserving functions, etc. So we see that $\mathbb{M}\mathsf{nd}(+,+)$ is isomorphic to the nerve of the category $\mathbb{\Delta}_+$ of finite ordinals and order-preserving maps.

Definition III.3.1 (the free homotopy coherent monad)**.** Let $\mathbb{M}\mathsf{nd}$ denote the quasi-categorically enriched category with a single object $+$ and whose endo-hom quasi-category

$$\mathbb{M}\mathsf{nd}(+,+) := \mathbb{\Delta}_+$$

is the nerve of the category of finite ordinals and order-preserving maps. Composition is given by the ordinal sum

$$\mathbb{M}\mathsf{nd}(+,+) \times \mathbb{M}\mathsf{nd}(+,+) \xrightarrow{\oplus} \mathbb{M}\mathsf{nd}(+,+).$$

Ignoring the nerve, we can think of $\mathbb{M}\mathsf{nd}$ as a strict 2-category. It has a universal property that is well-known:

Proposition III.3.2. *2-functors* $\mathbb{M}\mathsf{nd} \to \mathbb{C}\mathsf{at}$ *correspond to monads.*

Proof. A 2-functor $\mathbb{M}\mathsf{nd} \to \mathbb{C}\mathsf{at}$ picks out a category B as the image of $+$, and then defines a strictly monoidal functor $\mathbb{M}\mathsf{nd}(+,+) = \mathbb{\Delta}_+ \to \mathsf{B}^{\mathsf{B}}$. The category $\mathbb{\Delta}_+$ has a universal property: strictly monoidal functors out of $\mathbb{\Delta}_+$ correspond to monoids in the target, and a monad on B is just a monoid in the category B^{B} of endofunctors! $\square$

Thus reassured, we may define a notion of a homotopy coherent monad acting on an object in a quasi-categorically enriched category $\mathbb{K}$. On account of the examples listed in III.1.1, III.1.2, and III.1.4, we might think of the objects in a quasi-categorically enriched category as being "∞-categories" in some sense.

Definition III.3.3 ([RV2])**.** A **homotopy coherent monad** in a quasi-categorically enriched category $\mathbb{K}$ is a simplicial functor $\mathbb{M}\mathsf{nd} \to \mathbb{K}$ whose domain is the simplicial computad $\mathbb{M}\mathsf{nd}$. Explicitly, it picks out:

- an object $B \in \mathbb{K}$.
- a homotopy coherent diagram $\mathbb{\Delta}_+ \to \mathrm{Fun}(B,B)$ that is strictly monoidal with respect to composition. It sends the generating 0-arrow $t \colon + \to +$ to a 0-arrow $T \colon B \to B$ and identifies 1-arrows that assemble into a diagram

$$\begin{tikzcd} \mathrm{id}_B \arrow[r, "\eta"] & T \arrow[r, shift left=2, "\eta T"] \arrow[r, shift right=2, "T\eta"'] & T^2 \arrow[l, "\mu" description] \arrow[r, shift left=4] \arrow[r] \arrow[r, shift right=4] & T^3 \arrow[l, shift right=2] \arrow[l, shift left=2] & \cdots \end{tikzcd}$$

We interpret the simplicial functor $\mathbb{\Delta}_+ \to \mathrm{Fun}(B,B)$ defined by a homotopy coherent monad as being a homotopy coherent version of the monad resolution for (T, η, μ).

III.4. Homotopy coherent adjunctions

This definition of a homotopy coherent monad seems reasonable but are there any examples? One way to present a monad in classical category theory is via an adjunction.

Definition III.4.1. An **adjunction** in $\mathbb{C}\mathsf{at}$ is comprised of a pair of categories A and B together with functors $U\colon \mathsf{A} \to \mathsf{B}$ and $F\colon \mathsf{B} \to \mathsf{A}$ and natural transformations $\eta\colon \mathrm{id}_{\mathsf{B}} \to UF$ and $\epsilon\colon FU \to \mathrm{id}_{\mathsf{A}}$, called the **unit** and **counit** respectively, so that the diagrams

$$\begin{tikzcd} & FUF \arrow[dr, "\epsilon F"] & \\ F \arrow[ur, "F\eta"] \arrow[rr, equal] & & F \end{tikzcd} \qquad \begin{tikzcd} & UFU \arrow[dr, "U\epsilon"] & \\ U \arrow[ur, "\eta U"] \arrow[rr, equal] & & U \end{tikzcd} \tag{III.4.2}$$

commute.

Lemma III.4.3. *Any adjunction induces a monad $(UF, \eta, U\epsilon F)$ on B.*

Proof. An exercise in diagram chasing. □

There is a free-living 2-category $\mathbb{A}\mathsf{dj}$ containing an adjunction in the sense of a universal property analogous to Proposition III.3.2. It has two objects $+$ and $-$ and the four hom-categories displayed:

$$\begin{tikzcd} + \arrow[loop left, "\mathbb{\Delta}_+"] \arrow[r, bend left, "\mathbb{\Delta}_{-\infty}\cong\mathbb{\Delta}_{\infty}^{\mathrm{op}}"] \arrow[r, phantom, "\perp"] & - \arrow[l, bend left, "\mathbb{\Delta}_{\infty}\cong\mathbb{\Delta}_{-\infty}^{\mathrm{op}}"] \arrow[loop right, "\mathbb{\Delta}_+^{\mathrm{op}}"] \end{tikzcd}$$

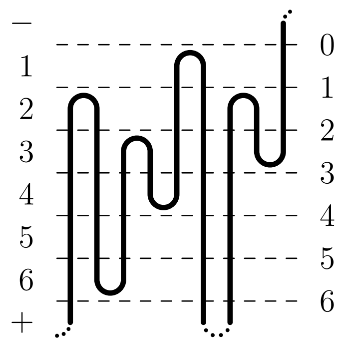
Here $\mathbb{\Delta}_\infty, \mathbb{\Delta}_{-\infty} \subset \mathbb{\Delta} \subset \mathbb{\Delta}_+$ are the subcategories of order-preserving maps that preserve the top or bottom elements, respectively, in each ordinal. The composition maps in $\mathbb{A}\mathsf{dj}$ are all restrictions of the ordinal sum operation.

Proposition III.4.4 ([SS])**.** *2-functors $\mathbb{A}\mathsf{dj} \to \mathbb{C}\mathsf{at}$ correspond to adjunctions in $\mathbb{C}\mathsf{at}$.*

We saw in Definition III.3.3 that the free homotopy coherent monad $\mathbb{M}\mathsf{nd}$ is in fact the free 2-category containing a monad: when this 2-category is regarded as a simplicial category by identifying its hom-category $\mathbb{\Delta}_+$ with its nerve, this simplicial category turns out to be a simplicial computad whose atomic n-arrows are those n-simplices whose final vertex is the 0-arrow t. The following result tells us that the same is true for adjunctions: the 2-category $\mathbb{A}\mathsf{dj}$, when regarded as a simplicial category, is a simplicial computad that defines the **free homotopy coherent adjunction**. Moreover, we present a convention graphical description of its n-arrows that establishes this simplicial computad structure.

Proposition III.4.5 ([RV2])**.** *The 2-category $\mathbb{A}\mathsf{dj}$, when regarded as a simplicial category via the nerve, is a simplicial computad with:*

- *two objects $+$ and $-$*
- *two atomic 0-arrows $f\colon + \to -$ and $u\colon - \to +$*
- *n-arrows given by strictly undulating squiggles on $(n+1)$-lines*



that are atomic if and only if there are no "instances of + or −" in their interiors.

Proof. An n-arrow lies in $\mathbb{A}\mathsf{dj}(-,+)$ if starts in the space labeled - on the right and ends in the space labeled + on the left; the description of the other hom simplicial sets is similar. The face and degeneracy maps act on the simplicial sets $\mathbb{A}\mathsf{dj}(+,+)$, $\mathbb{A}\mathsf{dj}(-,+)$, $\mathbb{A}\mathsf{dj}(+,-)$, and $\mathbb{A}\mathsf{dj}(+,+)$ by removing and duplicating lines. Composition is by horizontal juxtaposition, which makes the simplicial computad structure clear. □

Remark III.4.6. Note this gives a graphical calculus on the full subcategory $\mathbb{M}\mathsf{nd} \hookrightarrow \mathbb{A}\mathsf{dj}$. The n-arrows are strictly undulating squiggles on $(n+1)$-lines that start and end at the space labeled +; these are atomic if and only if there are no instances of + in their interiors. This condition implies that if all the lines are removed except the bottom one, a process that computes the final vertex of the n-simplex, the resulting squiggle looks like a single hump over one line, which is the graphical representation of the 0-arrow t. Because atomic arrows in $\mathbb{M}\mathsf{nd}$ may pass through $-$, $\mathbb{M}\mathsf{nd} \hookrightarrow \mathbb{A}\mathsf{dj}$ is not a subcomputad inclusion.

Definition III.4.7. A **homotopy coherent adjunction** in a quasi-categorically enriched category $\mathbb{K}$ is a simplicial functor $\mathbb{A}\mathsf{dj} \to \mathbb{K}$. Explicitly, it picks out:

- a pair of objects $A, B \in \mathbb{K}$.
- homotopy coherent diagrams

$$\mathbb{\Delta}_+ \to \mathrm{Fun}(B,B), \quad \mathbb{\Delta}_+^{\mathrm{op}} \to \mathrm{Fun}(A,A), \quad \mathbb{\Delta}_\infty \to \mathrm{Fun}(A,B), \quad \mathbb{\Delta}_\infty^{\mathrm{op}} \to \mathrm{Fun}(B,A)$$

  that are functorial with respect to the composition action of $\mathbb{A}\mathsf{dj}$.

The 0- and 1-dimensional data of the first and third of these may be depicted as follows

$$\begin{tikzcd}
\mathrm{id}_B \arrow[r, "\eta"] & UF \arrow[r, shift left=4, "\eta UF"] \arrow[r, shift right=4, "UF\eta"'] & UFUF \arrow[l, "U\epsilon F" description] \arrow[r, shift left=6] \arrow[r] \arrow[r, shift right=6] & UFUFUF \arrow[l, shift left=3] \arrow[l, shift right=3] & \cdots
\end{tikzcd}$$

$$\begin{tikzcd}
U \arrow[r, shift left=2, "\eta U"] & UFU \arrow[l, shift left=2, "U\epsilon"] \arrow[r, shift left=6, "\eta UF"] \arrow[r, shift right=2, "UF\eta" description] & UFUFU \arrow[l, shift right=2, "U\epsilon F" description] \arrow[l, shift left=6, "UFU\epsilon"'] \arrow[r, shift left=7.5] \arrow[r, shift left=1.5] \arrow[r, shift right=4.5] & UFUFUFU \arrow[l, shift left=4.5] \arrow[l, shift right=1.5] \arrow[l, shift right=7.5] & \cdots
\end{tikzcd}$$

the remaining two diagrams being dual. We interpret the homotopy coherent diagrams $\mathbb{\Delta}_+ \to \mathrm{Fun}(B,B)$, $\mathbb{\Delta}_+^{\mathrm{op}} \to \mathrm{Fun}(A,A)$, $\mathbb{\Delta}_\infty \to \mathrm{Fun}(A,B)$, and $\mathbb{\Delta}_\infty^{\mathrm{op}} \to \mathrm{Fun}(B,A)$ as defining homotopy coherent versions of the bar and cobar resolutions of the adjunction (F, U, η, ϵ).

Any homotopy coherent adjunction has an underlying adjunction in the following sense. A quasi-categorically enriched category $\mathbb{K}$ has an associated **homotopy 2-category** defined by applying the homotopy category functor Ho to each hom-category. Now, an **adjunction in a quasi-categorically enriched category** is an adjunction the homotopy 2-category obtained by taking the hom-categories of the function complexes. Explicitly, an adjunction is given by:

- a pair of objects A and B,
- a pair of 0-arrows $U\colon A \to B$ and $F\colon B \to A$,
- a pair of 1-arrows $\eta\colon \mathrm{id}_B \to UF$ and $\epsilon\colon FU \to \mathrm{id}_A$

so that there exist 2-arrows whose boundaries have the form displayed in (III.4.2).

The upshot is that an adjunction in a quasi-categorically enriched category is not so hard to define in practice and this low-dimensional data may be extended to give a full homotopy coherent adjunction:

Theorem III.4.8 ([RV2, 4.3.11,4.3.13])**.** *Any adjunction in a quasi-categorically enriched category extends to a homotopy coherent adjunction.*

Moreover extensions from judiciously chosen basic adjunction data are homotopically unique [RV2, §4.4].

Remark III.4.9. It is also fruitful to consider simplicial functors $\mathbb{A}\mathsf{dj} \to \mathbb{S}$ valued in a Kan complex enriched category. Because all 1-arrows in $\mathbb{S}$ are invertible, the unit and counit in this case are natural isomorphisms and this data is more properly referred to as a **homotopy coherent adjoint equivalence**. Theorem III.4.8 implies that any adjoint equivalence in a Kan complex enriched category extends to a homotopy coherent adjoint equivalence. Paired with the familiar 2-categorical result that says that any equivalence can be promoted to an adjoint equivalence, we conclude that any equivalence in a Kan complex enriched category extends to a homotopy coherent adjoint equivalence.

III.5. Algebras for a homotopy coherent monad

Finally, we connect these homotopy coherent notions to "algebra."

Definition III.5.1. Let (T, η, μ) be a monad acting on a category B. A T-**algebra** is a pair $(b, \beta\colon Tb \to b)$ so that

$$\begin{tikzcd} b \arrow[r, shift left, "\eta"] & Tb \arrow[l, shift left, "\beta"] \arrow[r, shift left=3, "\eta T"] \arrow[r, shift right=1, dashed, "T\eta"'] & T^2 b \arrow[l, shift right=1, dashed, "\mu"'] \arrow[l, shift left=3, "T\beta"] \end{tikzcd}$$

defines a truncated split augmented simplicial object.[20]

T-algebras in B and T-algebra homomorphisms define the **category of algebras**, traditionally denoted by B^T.

Proposition III.5.2. *Let (T, η, μ) be a monad acting on a category B. There is an adjunction*

$$\begin{tikzcd} \mathsf{B} \arrow[r, bend left, "F^T"] \arrow[r, phantom, "\perp"] & \mathsf{B}^T \arrow[l, bend left, "U^T"] \end{tikzcd}$$

[20]The shape of this diagram is given by the full subcategory of $\mathbb{\Delta}_\infty$ spanned by the objects [0], [1], and [2].

whose underlying monad is T.

If $B \in \mathbb{K}$ is thought of as an ∞-category and $\mathbb{M}\mathsf{nd} \to \mathbb{K}$ is a homotopy coherent monad on B, then the ∞-category of T-algebras in B may be recovered as an appropriately defined *flexible weighted limit* of the diagram $\mathbb{M}\mathsf{nd} \to \mathbb{K}$. This limit computes the value of the right Kan extension along $\mathbb{M}\mathsf{nd} \hookrightarrow \mathbb{A}\mathsf{dj}$ at the object $-$ and so in fact constructs the entire homotopy coherent adjunction. All of the examples of quasi-categorically enriched categories $\mathbb{K}$ mentioned above are ∞-*cosmoi*, in which such limits exist.

A full description of the ∞-category of algebras for a homotopy coherent monad is given in [RV2, §6], but we can at least give an informal description here. To a rough approximation, a *homotopy coherent T-algebra* for a homotopy coherent monad acting on an object $B \in \mathbb{K}$ is a homotopy coherent diagram of shape $\mathbb{\Delta}_\infty$ in B satisfying various functoriality conditions, which are suggested by the picture

$$\begin{tikzcd}
b \arrow[r, shift left, "\eta"] & Tb \arrow[l, shift left, "\beta"] \arrow[r, shift left=4.5, "\eta T"] \arrow[r, shift right=1.5, "T\eta" description] & T^2 b \arrow[l, shift right=1.5, "\mu" description] \arrow[l, shift left=4.5, "T\beta"] \arrow[r, shift left=5] \arrow[r, shift left=1] \arrow[r, shift right=3] & T^3 b \arrow[l, shift right=5] \arrow[l, shift right=1] \arrow[l, shift left=3] & \cdots
\end{tikzcd}$$

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