∞-CATEGORY THEORY FOR UNDERGRADUATES

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THESIS: If future undergraduates' foundational understanding of mathematical proof were based on Homotopy Type Theory (HoTT) then we could teach them ∞-category theory — much as we teach todays undergraduates abstract algebra.

ACT I: undergraduate-level informal HoTT   ACT II: ∞-category theory for undergraduates

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ACT I: undergraduate-level informal HoTT

Dependent type theory is a formal system of inference rules, that combine to form derivations

There are four kinds of "well-formed formulas" called judgments, including:

$\Gamma \vdash A$ type   "A is a type"

$\Gamma \vdash a : A$ "a is a term of type A"

Here "Γ" is a context which declares the types of any variables that appear: eg

$\Gamma, x : A \vdash B(x)$ type   "a family of types over A" $n : \mathbb{N} \vdash \mathbb{R}^n$ type

$\Gamma, x : A \vdash b(x) : B(x)$ "a family of terms" $n : \mathbb{N} \vdash \vec{0} : \mathbb{R}^n$

There are four kinds of rules (in place of axioms) that can be used in derivations:

(i) formation rules form new types:

×formation: given types A and B there is a product type $A \times B$

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type}}{\Gamma \vdash A \times B \text{ type}}$$

(ii) introduction rules introduce new terms:

×introduction: given terms $a : A$ and $b : B$ there is a term $(a,b) : A \times B$

$$\frac{\Gamma \vdash a : A \quad \Gamma \vdash b : B}{\Gamma \vdash (a,b) : A \times B}$$

(iii) elimination rules use the new terms:

×elimination: given a term $p : A \times B$ there are terms $pr_1(p) : A$ and $pr_2(p) : B$

$$\frac{\Gamma \vdash p : A \times B}{\Gamma \vdash pr_1(p) : A} \qquad \frac{\Gamma \vdash p : A \times B}{\Gamma \vdash pr_2(p) : B}$$

(iv) Computation rules relate (ii) and (iii)

Function types are governed by the rules:

$\to$formation: given types A and B, there is a type $A \to B$

$\to$introduction: if in the context of any term $x : A$ there is a term $b(x) : B$, then there is a term $\lambda x. b(x) : A \to B$

$$\frac{\Gamma, x : A \vdash b(x) : B}{\Gamma \vdash \lambda x. b(x) : A \to B}$$

$\to$elimination: given terms $f : A \to B$ and $a : A$, there is a term $f(a) : B$

+ two computation rules.

A proposition is proven by constructing a term in the type that encodes its statement.

Proposition: For any types P and Q, there is a term $\text{modus-ponens} : P \times (P \to Q) \to Q$.

Proof: By $\to$introduction we must explain how to use a term $x : P \times (P \to Q)$ to produce a term of type Q. By $\times$elimination from x we get terms $pr_1(x) : P$ and $pr_2(x) : P \to Q$. By $\to$elimination then $(pr_2(x))(pr_1(x)) : Q$. Ie, $\text{modus-ponens} :\equiv \lambda x. (pr_2(x)(pr_1(x)))$. $\square$

Propositions concerning mathematical equality are governed by Per Martin-Löf's identity types:

$=$formation: given a type A and two terms $x, y : A$, there is a type $x =_A y$

$=$introduction: given a term $x : A$, there is a term $\text{refl}_x : x =_A x$

The elimination rule for the identity type can be packaged into the principal of path induction:

Path induction: Given any type family $\Gamma, x, y : A, p : x =_A y \vdash B(x, y, p)$ type, to produce a term of type $B(x, y, p)$ it suffices to assume y is x and p is refl_x.

Lemma: For any $x, y : A$, $(x =_A y) \to (y =_A x)$.

Proof: By $\to$introduction, we may assume $p : x =_A y$, and must produce a term of type $y =_A x$. By path induction, to inhabit the type family $B(x, y, p) :\equiv y =_A x$, it suffices to assume y is x and p is refl_x, in which case by $=$introduction we have $\text{refl}_x : x =_A x$. $\square$

Lemma: For any $x, y, z : A$, $(x =_A y) \to ((y =_A z) \to (x =_A z))$.

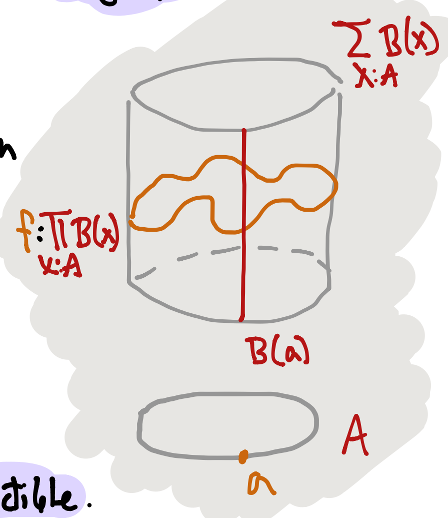
Proof: By $\to$introduction, we may assume $p : x =_A y$ and $q : y =_A z$ and seek to inhabit $x =_A z$. By path induction on p and then on q, we may assume y and z are x and p and q are refl_x in which case by $=$introduction we have $\text{refl}_x : x =_A x$. $\square$

The name "path induction" derives from the homotopical interpretation of dependent type theory.

$$\begin{cases} \text{a type } A \rightsquigarrow \text{a "space" } A \\ \text{a term } a : A \rightsquigarrow \text{a point } a \text{ in } A \\ \text{a term } p : x =_A y \rightsquigarrow \text{a path } p \text{ from } x \text{ to } y \text{ in } A \\ \text{a term } h : p \underset{x =_A y}{=} q \rightsquigarrow \text{a homotopy } h \text{ from } p \text{ to } q \text{ in } A \end{cases}$$

From this point of view, symmetry and transitivity of equality becomes reversal and composition of paths, and of homotopies, and of higher homotopies, as summarized by a theorem of Lumsdaine and van den Berg-Garner: types inherit the structure of an ∞-groupoid.

$$\begin{cases} \text{a type family } x : A \vdash B(x) \text{ type} \rightsquigarrow \text{a fibration over } A \\ \text{the dependent sum type } \sum_{x:A} B(x) \rightsquigarrow \text{the total space of a fibration} \\ \text{the dependent function type } \prod_{x:A} B(x) \rightsquigarrow \text{the space of sections} \end{cases}$$



The homotopical interpretation inspired the following definitions:

defn: There exists a unique term of type A just when the type

$$\sum_{a:A} \prod_{x:A} a =_A x$$

is inhabited, ie, just when the "space" A is contractible.

Path induction expresses the contractibility of based path spaces!

defn: Types A and B are equivalent just when the following type is inhabited:

$$A \simeq B :\equiv \sum_{f : A \to B} \left(\sum_{g : B \to A} \prod_{a:A} g(f(a)) =_A a \right) \times \left(\sum_{h : B \to A} \prod_{b:B} f(h(b)) =_B b \right).$$

By the elimination rules for dependent sums and functions, a term in $A \simeq B$ gives terms $f : A \to B$ and $g, h : B \to A$ together with homotopies $\alpha : \prod_{a:A} g(f(a)) =_A a$ and $\beta : \prod_{b:B} f(h(b)) =_B b$.

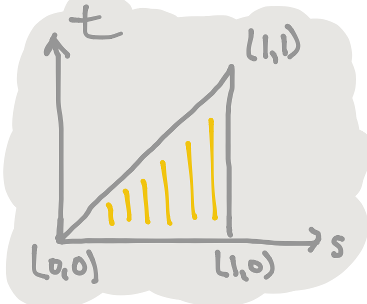
By composing these one can show that $\prod_{b:B} g(b) =_B h(b)$. But there's a good reason to define an equivalence to be a function $f : A \to B$ equipped with a priori distinct left and right inverses: given any $x, y : \left(\sum_{g : B \to A} \prod_{a:A} g(f(a)) =_A a \right) \times \left(\sum_{h : B \to A} \prod_{b:B} f(h(b)) =_B b \right)$ then $x = y$,

while the type $\sum_{g : B \to A} \left(\prod_{a:A} g(f(a)) =_A a \right) \times \left(\prod_{b:B} f(g(b)) =_B b \right)$ might have distinct terms.

ACT II: ∞-category theory for undergrads (joint with Mike Shulman)

We work in an extension of HoTT in which types are allowed to depend on polytopes within **directed cubes**: products of a directed interval 2, which has $0,1 : 2$ and $x,y : 2 \vdash x \leq y$

$$\Delta^n :\equiv \{\langle t_1,\ldots,t_n\rangle : 2^n \mid t_n \leq \cdots \leq t_1\}$$
$$\partial\Delta^2 :\equiv \{\langle s,t\rangle : 2^2 \mid (t \leq s) \wedge ((t=0) \vee (t=s) \vee (s=1))\}$$
$$\Lambda^2_1 :\equiv \{\langle s,t\rangle : 2^2 \mid (t \leq s) \wedge ((t=0) \vee (s=1))\}$$



Given polytopes $\Phi \subset \Psi$ and a function $f : \Phi \to A$ we may form an **extension type**:

$$\left\langle \begin{tikzcd} \Phi \arrow[r, "f"] \arrow[d, hook] & A \\ \Psi \arrow[ur, dashed] \end{tikzcd} \right\rangle$$ whose terms are $g : \Psi \to A$ so that $g|_\Phi \equiv f$.

Confidential to grad students: Semantics in Reedy fibrant bisimplicial sets

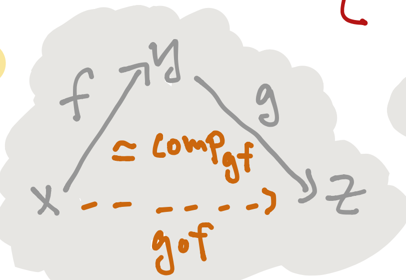
defn: Given $x,y : A$, $\hom_A(x,y) :\equiv \left\{ \begin{tikzcd} \partial\Delta^1 \arrow[r, "x{,}y"] \arrow[d, hook] & A \\ \Delta^1 \arrow[ur, dashed] \end{tikzcd} \right\}$ is the type of **arrows** in A from x to y.

defn: A type A is an **∞-groupoid** if path-to-arr : $x =_A y \to \hom_A(x,y)$ is an equivalence.

$\mathrm{refl}_x \mapsto \mathrm{id}_x$

defn: A type A is a **pre-∞-category** if every composable pair of arrows has a **unique composite**: for all $f : \hom_A(x,y)$ and $g : \hom_A(y,z)$ the type $\left\{ \begin{tikzcd} \Lambda^2_1 \arrow[r, "f{,}g"] \arrow[d, hook] & A \\ \Delta^2 \arrow[ur, dashed] \end{tikzcd} \right\}$ is contractible.

Notation: Denote the **unique** inhabitant by:

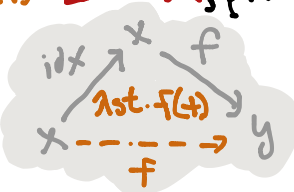


In any **pre ∞-category** A:

Lemma: Each $x : A$ has an **identity arrow** $\mathrm{id}_x : \hom_A(x,x)$ so that for all $f : \hom_A(x,y)$ and all $k : \hom_A(w,x)$
$f \circ \mathrm{id}_x = f$ and $\mathrm{id}_x \circ k = k$.

Proof: The constant function defines a term $\mathrm{id}_x :\equiv \lambda t.x : \hom_A(x,x) :\equiv \left\{ \begin{tikzcd} \partial\Delta^1 \arrow[r, "x{,}x"] \arrow[d, hook] & A \\ \Delta^1 \arrow[ur, dashed] \end{tikzcd} \right\}$

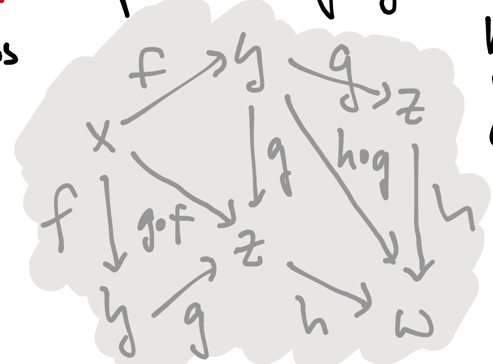
The type $\left\{ \begin{tikzcd} \Lambda^2_1 \arrow[r, "\mathrm{id}_x{,}f"] \arrow[d, hook] & A \\ \Delta^2 \arrow[ur, dashed] \end{tikzcd} \right\}$ is inhabited by $\lambda s \geq t, f(t) : \Delta^2 \to A$, proving $f \circ \mathrm{id}_x = f$.



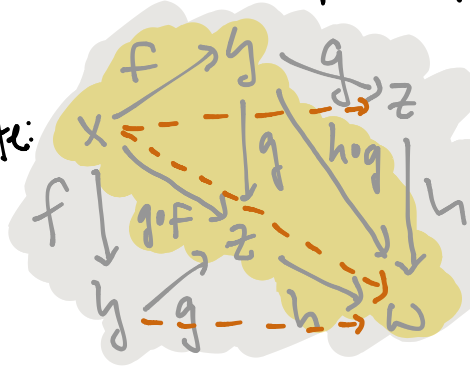
$\square$

Lemma: Composition is **associative**: for all $f : hom_A(x,y)$, $g : hom_A(y,z)$, and $h : hom_A(z,w)$
$$h\circ(g\circ f) = (h\circ g)\circ f.$$

Proof: If A is a pre ∞-category so is $\Delta^1 \to A$ so the composable pair of arrows



has a unique composite:



$\square$

defn: An arrow $f : hom_A(x,y)$ in a pre ∞-category is an **isomorphism** if it has left and right composition inverses:

$$x \cong_A y :\equiv \sum_{f : hom_A(x,y)} \left(\sum_{g : hom_A(y,x)} g\circ f = id_x \right) \times \left(\sum_{h : hom_A(y,x)} f \circ h = id_y \right)$$

Exercise: prove then that $g = h$

Recall a type A is an **∞-groupoid** if for all $x, y : A$, path-to-arr $: x =_A y \to hom_A(x,y)$ is an equivalence.

$refl_x \mapsto id_x$

defn: A type A is an **∞-category** if

- every composable pair of arrows has a **unique composite**: ie,
$$\left\{ \begin{tikzcd} \Lambda^2_1 \arrow[r, "f{,}g"] \arrow[d] & A \\ \Delta^2 \arrow[ur, dashed] & \end{tikzcd} \right\}$$
is contractible.
- isomorphisms are equivalent to identities:

for all $x, y : A$, path-to-iso $: x =_A y \to x \cong_A y$ is an equivalence

$refl_x \mapsto id_x$

Theorem: A is an **∞-groupoid** iff A is an **∞-category** and all of its arrows are **isomorphisms**.

Proof: In the diagram
$$\begin{tikzcd} x =_A y \arrow[r, "\text{path-to-arr}"] \arrow[dr, "\text{path-to-iso}"'] & hom_A(x,y) \\ & x \cong_A y \arrow[u, hook] \end{tikzcd}$$
the inclusion is an equivalence iff its surjective, ie iff all arrows are isomorphisms. If A is an ∞-category and all of its arrows are isos, these equivalences compose. If A is an ∞-groupoid path-to-arr is surjective, so $x \cong_A y \hookrightarrow hom_A(x,y)$ is an equivalence, so path-to-iso is too. $\square$

EPILOGUE: A better Yoneda lemma

Path induction: Given any type family $\Gamma, x, y : A, p : x =_A y \vdash B(x,y,p)$ type, to produce a term of type $B(x,y,p)$ it suffices to assume y is x and p is refl_x.

a categorical fibration

Arrow induction: Given a pre ∞-category A and a covariantly functorial type family $\Gamma, x, y : A, f : \hom_A(x,y) \vdash B(x,y,f)$ type, to produce a term of type $B(x,y,f)$ it suffices to assume y is x and f is id_x.

covariant in y and f

Yoneda Lemma: Given a pre ∞-category A, a term $a : A$, and a covariantly functorial type family $\Gamma, x : A \vdash B(x)$ type, the function $\mathrm{ev\text{-}id}_a :\equiv \lambda\alpha.\alpha(a, \mathrm{id}_a) : \left(\prod_{x:A} \left(\hom_A(a,x) \to B(x)\right)\right) \to B(a)$ is an equivalence.

Corollary: For any a and b in a pre-∞-category A if $\prod_{x:A} \hom_A(a,x) \simeq \hom_A(b,x)$ then $a \cong_A b$ and if A is an ∞-category then $a =_A b$.

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