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On the directed univalence axiom

joint with Evan Cavallo and Christian Sattler

AMS Special Session on Homotopy Type Theory, Joint Mathematics Meetings

Plan

1. A type theory for synthetic $(\infty, 1)$-categories

2. A directed univalence conjecture

3. Covariant type families

4. The covariant directed univalence axiom

1

A type theory for synthetic $(\infty, 1)$-categories

The bisimplicial sets model

$\mathcal{S}et^{\Delta^{op} \times \Delta^{op}}$	$\supset$	$\mathcal{R}eedy$	$\supset$	$\mathcal{S}egal$	$\supset$	$\mathcal{R}ezk$
$\parallel$		$\parallel$		$\parallel$		$\parallel$
bisimplicial sets		types		types with composition		types with composition & univalence

Theorem (Shulman). Homotopy type theory is modeled by the category of Reedy fibrant bisimplicial sets.

Theorem (Rezk). $(\infty, 1)$-categories are modeled by Rezk spaces aka complete Segal spaces.

The bisimplicial sets model of homotopy type theory has:

- an interval type I, parametrizing paths inside a general type
- a directed interval type $\mathbb{2}$, parametrizing arrows inside a general type

Paths and arrows

- The identity type for A depends on two terms in A:

$$x, y : A \vdash x =_A y$$

  and a term $p : x =_A y$ may be thought of as a path in A from x to y.
- The hom type for A depends on two terms in A:

$$x, y : A \vdash \mathbf{hom}_A(x, y)$$

  and a term $f : \mathbf{hom}_A(x, y)$ defines an arrow in A from x to y.

Hom types are defined as instances of extension types axiomatized in a three-layered type theory with shapes due to Shulman

$$\mathbf{hom}_A(x, y) :\equiv \left\langle \begin{array}{ccc} 1+1 & \xrightarrow{[x,y]} & A \\ \downarrow & \nearrow & \\ 2 & & \end{array} \right\rangle$$

Segal, Rezk, and discrete types

- A type A is Segal if every composable pair of arrows has a unique composite: if for every $f : \mathbf{hom}_A(x, y)$ and $g : \mathbf{hom}_A(y, z)$

$$\left\langle \begin{array}{ccc} \Lambda^2_1 & \xrightarrow{[f,g]} & A \\ \rightarrowtail & \nearrow & \\ \Delta^2 & & \end{array} \right\rangle \qquad \text{is contractible.}$$

- A Segal type A is Rezk if every isomorphism is an identity: if

$$\text{id-to-iso} : \prod_{x,y:A} (x =_A y) \to (x \cong_A y) \qquad \text{is an equivalence.}$$

- A type A is discrete if every arrow is an identity: if

$$\text{id-to-arr} : \prod_{x,y:A} (x =_A y) \to \mathbf{hom}_A(x, y) \qquad \text{is an equivalence.}$$

Prop. A type is discrete if and only if it is Rezk and all of its arrows are isomorphisms — the discrete types are the ∞-groupoids.

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A directed univalence conjecture

What are the arrows in the universe?

For small types $A, B : \mathcal{U}$, the following are equivalent:

- an arrow $F : \mathbf{hom}_{\mathcal{U}}(A, B)$
- a function $F : \mathbb{2} \to \mathcal{U}$ with $F(0) \equiv A$ and $F(1) \equiv B$
- a type family $t : \mathbb{2} \vdash F(t)$ with $F(0) \equiv A$ and $F(1) \equiv B$

In this context the dependent function type is equivalent to the dependent sum

$$\prod_{t:\mathbb{2}} F(t) \simeq \sum_{a:A} \sum_{b:B} \mathbf{hom}_{F(\mathbb{2})}(a, b)$$

of dependent hom types

$$\mathbf{hom}_{F(\mathbb{2})}(a, b) :\equiv \left\langle \begin{array}{ccc} & \overset{[a,b]}{\nearrow} & F(\mathbb{2}) \\ & & \downarrow \;\; \uparrow \\ 1 + 1 & \rightarrowtail & \mathbb{2} \end{array} \right\rangle,$$

the type of arrows in F from a to b over the generic arrow in $\mathbb{2}$.

A conjectural directed univalence axiom

Define

$$\text{arr-to-span} : \mathbf{hom}_{\mathcal{U}}(A, B) \to (A \times B \to \mathcal{U})$$

to carry F to the span given by the dependent product

$$A \xleftarrow{\text{dom}} \prod_{\mathbb{2}} F \simeq \sum_{a:A} \sum_{b:B} \mathbf{hom}_{F(\mathbb{2})}(a, b) \xrightarrow{\text{cod}} B$$

and its domain and codomain projections.

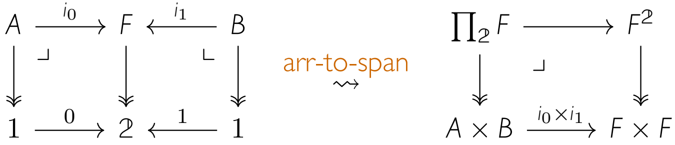
Directed Univalence Conjecture.
For all small types A and B the map

$$\text{arr-to-span} : \mathbf{hom}_{\mathcal{U}}(A, B) \to (A \times B \to \mathcal{U})$$

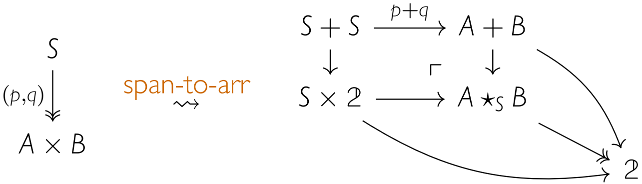
is an equivalence.

Semantics of the directed univalence conjecture

Semantically, arr-to-span constructs the comma object of a cospan:



2-category theory suggests a converse construction:



The image of arr-to-span is not all spans — only the "two-sided discrete fibrations" — the definition of which involves conditions on A and B.

⇝ Search for a directed univalence axiom in a different universe.

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Covariant type families

Covariant type families I

Let $x : A \vdash B(x)$ be a type family over a Segal type A. Then any arrow $f : \mathbf{hom}_A(x, y)$ in the base, gives rise to a span

$$B(x) \xleftarrow{\text{dom}} \sum_{u:B(x)} \sum_{v:B(y)} \mathbf{hom}_{B(f)}(u, v) \xrightarrow{\text{cod}} B(y)$$

and any 2-simplex in A witnessing $h = g \circ f$ gives rise to a "higher span."

A type family $x : A \vdash B(x)$ over a Segal type A is covariant if for every $f : \mathbf{hom}_A(x, y)$ and $u : B(x)$ there is a unique lift of f with domain u., i.e.:

$$\sum_{v:B(y)} \mathbf{hom}_{B(f)}(u, v) \quad \text{is contractible.}$$

$x : A \vdash B(x)$ is covariant iff for each $f : \mathbf{hom}_A(x, y)$ the left leg of the span from $B(x)$ to $B(y)$ is an equivalence — defining a covariant span.

Covariant type families II

A type family $x : A \vdash B(x)$ over a Segal type A is covariant if for every $f : \mathbf{hom}_A(x, y)$ and $u : B(x)$ there is a unique lift of f with domain u.

Prop. If $x : A \vdash B(x)$ is covariant then for each $x : A$ the fiber $B(x)$ is discrete. Thus covariant type families are fibered in ∞-groupoids.

Prop. Fix $a : A$. The type family $x : A \vdash \mathbf{hom}_A(a, x)$ is covariant.

The Yoneda lemma proves that the type family $x : A \vdash \mathbf{hom}_A(a, x)$ is freely generated by the identity arrow $\mathrm{id}_a : \mathbf{hom}_A(a, a)$ and gives a "directed" version of the "transport" operation for identity types.

The universe of covariant fibrations

In bisimplicial sets

- type families correspond to Reedy fibrations, characterized by a right lifting property against:

$$(\partial\Delta^m \to \Delta^m)\widehat{\square}(\Lambda^n_k \to \Delta^n) \qquad m \geq 0,\ 0 \leq k \leq n$$

- covariant type families correspond to covariant fibrations aka left fibrations, characterized by a further right lifting property against:

$$(\Lambda^n_k \to \Delta^n)\widehat{\square}(\partial\Delta^m \to \Delta^m) \qquad m \geq 0,\ 0 \leq k < n.$$

The universe of covariant fibrations $\mathcal{U}_{\mathsf{cov}}$ is the presheaf on $\mathbb{\Delta} \times \mathbb{\Delta}$ with

$$\mathcal{U}_{\mathsf{cov}}(m,n) := \{\text{covariant fibrations over } \Delta^m \square \Delta^n\}.$$

The universal covariant fibration is defined by pullback:

$$\begin{array}{ccc} \widetilde{\mathcal{U}}_{\mathsf{cov}} & \longrightarrow & \widetilde{\mathcal{U}} \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{U}_{\mathsf{cov}} & \longrightarrow & \mathcal{U} \end{array}$$

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The covariant directed univalence axiom

A new directed univalence axiom

- A covariant type family over $\mathbb{1}$ is a discrete type. Thus the terms in $\mathcal{U}_{\mathsf{cov}}$ are discrete types.
- A covariant type family $t : \mathbb{2} \vdash F(t)$ over $\mathbb{2}$ determines a pair of discrete types $A := F(0)$ and $B := F(1)$ together with a span

$$A \xleftarrow{\ \mathsf{dom}\ } \sum_{a:A} \sum_{b:B} \mathbf{hom}_{F(\mathbb{2})}(a, b) \xrightarrow{\ \mathsf{cod}\ } B$$

whose left leg is invertible. The type of such covariant spans is equivalent to the type of functions $A \to B$.

Directed Univalence Axiom. For all small discrete types A and B the map

$$\mathsf{arr\text{-}to\text{-}fun} : \mathbf{hom}_{\mathcal{U}_{\mathsf{cov}}}(A, B) \to (A \to B)$$

is an equivalence.

Evidence supporting the directed univalence axiom

Directed Univalence Axiom. For all small discrete types A and B the map

$$\text{arr-to-fun} : \mathbf{hom}_{\mathcal{U}_{\text{cov}}}(A, B) \to (A \to B)$$

is an equivalence.

Sattler has sketched a verification of the Directed Univalence Axiom in bisimplicial sets:

- The canonical map $\mathcal{U}_{\text{cov}} \to \mathcal{U}$ is a fibration; hence $\mathcal{U}_{\text{cov}}$ is fibrant.
- The homotopy inverse to arr-to-fun is the specialization of span-to-arr to the case of covariant spans between discrete types.
- This map cov-span-to-arr automatically produces a covariant fibration over $\mathbb{2}$.
- The fatal flaw in the original directed univalence conjecture is avoided since discrete types are local at $\mathbb{2}$: $A \simeq I \to A \simeq \mathbb{2} \to A$.

A warning about the universal property of $\mathcal{U}_{cov}$

The type theoretic definition of a covariant type family can be stated in any context and the universe for covariant fibrations $\mathcal{U}_{cov}$ can be weakened to any context.

- A covariant type family $x : A \vdash B(x)$ over A in the empty context defines a map $B : A \to \mathcal{U}_{cov}$ and conversely.
- But a covariant type family $x : A \vdash B(x)$ over A in context Γ will not define a map $B : \Gamma.A \to \mathcal{U}_{cov}$.
- The definition of a covariant type family over A in context Γ is covariant over arrows in A fiberwise in Γ.
- Whereas a map $B : \Gamma.A \to \mathcal{U}_{cov}$ defines a type family that is covariant over arrows in the entire extended context.

Summary

- A type theory for synthetic $(\infty, 1)$-categories with semantics in the bisimplicial sets model of HoTT has been developed by Riehl–Shulman but many questions about universes remain.
- A directed univalence conjecture — that arrows in the universe of all types are equivalent to spans — is false in the model.
- A restricted directed univalence axiom — that arrows in the universe of covariant fibrations correspond to functions between discrete types — is likely true in the model.
- Much remains to be explored, so let us know if you'd like to get involved!

References

For considerably more, see:

Emily Riehl and Michael Shulman, A type theory for synthetic ∞-categories, Higher Structures 1(1):116–193, 2017. arXiv:1705.07442

Michael Shulman, The univalence axiom for elegant Reedy presheaves, Homology, Homotopy, and Applications, 17(2):81–106, 2015. arXiv:1307.6248

Thank you!