MATH 616: ALGEBRAIC TOPOLOGY: THE MODEL-INDEPENDENT THEORY OF ∞-CATEGORIES

EMILY RIEHL

Logistics

Lectures: MW 1:30-2:45, Gilman 219

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Office hours: immediately following each lecture, or by appointment.

Content

Mathematical objects — such as spaces, chain complexes, spectra, schemes — that have a positive integer indexed hierarchy of "higher-dimensional" homomorphisms live most naturally in infinite-dimensional categories, nicknamed ∞-categories. Unlike the case for ordinary categories, composition of morphisms in an ∞-category is seldom strictly associative or unital, but instead satisfies these axioms up to the presence of a higher-dimensional invertible morphism. The infinite-dimensional categories we will consider are more accurately referred to as $(\infty, 1)$-categories, the index signaling that all morphisms above dimension 1 are invertible in the weak sense just described.

∞-categories provide a convenient framework to study the homotopical[1] behavior of their objects. For instance, the ∞-category of chain complexes gives a presentation of the homotopy theory of chain complexes that is more powerful than the derived category of chain complexes up to quasi-isomorphism. The derived category is "triangulated," with the triangles representing long exact sequences, but certain constructions (eg the cone of a chain map) that can be built from this triangulated structure fail to define derived functors because they require higher homotopical information that has been lost. By contrast, at the level of the ∞-category of chain complexes, the "triangulated structure" present on the derived category is encoded as a short list of properties of the associated stable ∞-category, and the mapping cone construction is functorial at this level. This is typical: derived constructions frequently define honest functors between ∞-categories.

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[1]Meaning up to equivalence rather than up to isomorphism.

André Joyal and Jacob Lurie have proven that ∞-categories can be manipulated much like ordinary categories, but the precise ∞-categorical analogs of 1-categorical theorems can be complicated to state and tricky to prove. There is arguably no intrinsic reason for this added difficulty; instead it is an artifice of the fact that a precise construction of an $(\infty, 1)$-category in and of itself is quite complex, with a variety of different "models" of this schematically-defined notion encoding these infinite dimensional categories as simplicial sets, simplicial spaces, marked simplicial sets, and so on.

This course will introduce a pioneering new approach to ∞-category theory that eliminates the focus on the combinatorial details of any particular model of ∞-categories and instead makes use of an axiomatized list of properties of the surrounding ∞-cosmos, in which they live. From these axioms, we can define adjunctions and equivalences between ∞-categories, limits and colimits of diagrams valued in ∞-categories, and Kan extensions along functors between ∞-categories and prove the expected categorical theorems that relate these notions. When we specialize to the model considered by Joyal and Lurie, we recover the theory of ∞-categories that they developed, but our framework specializes to other models besides and can be used to prove that the formal theory of ∞-categories is "model independent" in a strong sense.

The style of proofs that are used are familiar to those that would appear in a more traditional second-semester Math 616: Algebraic Topology course (i.e., in homotopical algebra as encoded by a Quillen model category) or in category theory. For those comfortable with some level of abstract nonsense, they generally proceed along reassuringly familiar lines.

Prerequisites

To learn about ∞-categories, students will need prior familiarity with standard category theory and with the category of simplicial sets. Some self-study of these topics between now and the start of term is encouraged. A prerequisite problem set has been posted on the course website to guide students through these topics. This problem set is "moral homework" and will not be graded, but I'm happy to discuss solutions to the problems at any point.

Graduate students in mathematics are welcome to enroll in this course if they feel they have sufficient prerequisites. Other students are welcome to enroll with permission of the instructor or audit on their own prerogative.

Text

The primary course text will be a textbook, co-written with Dominic Verity and tentative entitled Elements of ∞-Category Theory, that will be drafted as we progress.

For a preview of the material that we will discuss over the course of the semester, see "∞-category theory from scratch," lecture notes written to accompany a four-hour mini course on these topics.

Background category theory can be found in Category Theory in Context or any text that the reader prefers. Background on simplicial sets can be gathered by osmosis or from any introductory text of the reader's choosing.

Assessment

The homework assignments for this course will be optional and take two forms.

- Periodically, problem sheets will be posted to the course website. A week or so later, we will schedule a problem session, to be held in my office, during which the solutions will be discussed.
- Any student who wishes can write an optional expository final paper, with a target length of 5-10 pages, presenting a topic in ∞-category theory that we will not have time to explore thoroughly in class. Students who are interested in writing a paper should approach me around Spring Break to discuss possible topics.

Grades will only be assigned if absolutely necessary, in which case they will be based on course participation.

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