Math 616: Algebraic Topology
Prerequisite problem set[1]
due: never

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A **simplicial set** is a contravariant Set-valued functor indexed by $\mathbb{\Delta}$, the category of finite non-empty ordinals $[n] = \{0 < 1 < \cdots < n\}$ and order-preserving maps.

Exercise 1. Prove that the category of simplicial sets — or any category of the form $\mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}$ — is cartesian closed.

The standard n-simplex $\Delta[n]$ is the simplicial set represented by the ordinal $[n] \in \mathbb{\Delta}$. As a simplicial set, it is n-**skeletal**, with only degenerate simplices in dimension greater than n. Informally, the n-skeletal simplicial sets "have dimension n."

Exercise 2.
(i) What is the dimension of $\Delta[n] \times \Delta[m]$?
(ii) A top-dimensional non-degenerate simplex of the simplicial set $\Delta[m] \times \Delta[n]$ is called a *shuffle*. Explain which shuffles share codimension-one faces and use this to give a sensible partial ordering on the set of shuffles with a minimal and a maximal element.[2]

Exercise 3. Explain the sense in which the boundary inclusions $\{\partial\Delta[n] \hookrightarrow \Delta[n]\}_{n \geq 0}$ generate the monomorphisms in the category of simplicial sets.

Exercise 4. Let $\otimes \colon \mathsf{A} \times \mathsf{B} \to \mathsf{C}$ be a functor of two variables admitting a parametrized right adjoint $\{\} \colon \mathsf{A}^{\mathrm{op}} \times \mathsf{C} \to \mathsf{B}$, i.e., so that

$$\mathsf{C}(A \otimes B, C) \cong \mathsf{B}(B, \{A, C\})$$

naturally in A, B, and C.[3] Assuming C has pushouts and B has pullbacks define bifunctors $\hat{\otimes} \colon \mathsf{A}^{\mathbf{2}} \times \mathsf{B}^{\mathbf{2}} \to \mathsf{C}^{\mathbf{2}}$ and $\widehat{\{\}} \colon (\mathsf{A}^{\mathbf{2}})^{\mathrm{op}} \times \mathsf{C}^{\mathbf{2}} \to \mathsf{B}^{\mathbf{2}}$ between the categories of arrows and commutative squares in A, B, and C as follows:

$$\begin{tikzcd}
A \otimes B \arrow[r, "A \otimes b"] \arrow[d, "a \otimes B"'] & A \otimes B' \arrow[d] \arrow[ddr, bend left, "a \otimes B'"] & \\
A' \otimes B \arrow[r] \arrow[drr, bend right, "A' \otimes b"'] & \bullet \arrow[ul, phantom, "\ulcorner" very near start] \arrow[dr, dashed, "a \hat{\otimes} b"] & \\
 & & A' \otimes B'
\end{tikzcd}
\qquad
\begin{tikzcd}
\{A', C\} \arrow[drr, bend left, "\{a,C\}"] \arrow[dr, dashed, "\widehat{\{a,c\}}"] \arrow[ddr, bend right, "\{A',c\}"'] & & \\
 & \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & \{A, C\} \arrow[d, "\{A,c\}"] \\
 & \{A', C'\} \arrow[r, "\{a,C'\}"'] & \{A, C'\}
\end{tikzcd}$$

Prove that $\hat{\otimes} \colon \mathsf{A}^{\mathbf{2}} \times \mathsf{B}^{\mathbf{2}} \to \mathsf{C}^{\mathbf{2}}$ has $\widehat{\{\}} \colon (\mathsf{A}^{\mathbf{2}})^{\mathrm{op}} \times \mathsf{C}^{\mathbf{2}} \to \mathsf{B}^{\mathbf{2}}$ as its parametric right adjoint by defining a natural bijective correspondence between squares on the right

[1]These exercises are meant to guide you through the background literature on simplicial sets and category theory. I'm expecting most students will have to look various things up in order to solve these exercises.

[2]As far as I can tell, there is no reason to prefer this ordering to its opposite one.

[3]In a cartesian closed category C, the product $\times \colon \mathsf{C} \times \mathsf{C} \to \mathsf{C}$ has such an adjoint; see Exercise 1.

and squares on the left

$$\begin{tikzcd} A' \otimes B \cup_{A\otimes B} A \otimes B' \arrow[r, "(f^\sharp, g^\sharp)"] \arrow[d, "a \hat{\otimes} b"'] & C \arrow[d, "c"] \\ A' \otimes B' \arrow[r, "h^\sharp"'] \arrow[ur, dashed, "\ell^\sharp"] & C' \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "f^\flat"] \arrow[d, "b"'] & \{A', C\} \arrow[d, "\widehat{\{a,c\}}"] \\ B' \arrow[r, "(g^\flat, h^\flat)"'] \arrow[ur, dashed, "\ell^\flat"] & \{A', C'\} \times_{\{A, C'\}} \{A, C\} \end{tikzcd}$$

Observe that solutions to lifting problems, as represented by the dashed diagonal morphisms, also transpose.

Exercise 5. Consider a pair of horizontally composable natural transformations

$$\begin{tikzcd} \mathsf{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathsf{B} \arrow[r, bend left, "H"] \arrow[r, bend right, "K"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathsf{C} \end{tikzcd}$$

Express their horizontal composite

$$\begin{tikzcd} \mathsf{A} \arrow[r, bend left, "HF"] \arrow[r, bend right, "KG"'] \arrow[r, phantom, "\Downarrow\beta * \alpha"] & \mathsf{C} \end{tikzcd}$$

as a vertical composite of whiskered copies of α and β in two different ways and explain how this relationship may be expressed as a commutative diagram in the category C^{A} of functors and natural transformations from A to C.

Exercise 6.

(i) Given any functors $\mathsf{B} \xrightarrow{F} \mathsf{A} \xleftarrow{G} \mathsf{C}$, define a category $F \downarrow G$, two "projection functors" $\mathsf{B} \twoheadleftarrow F \downarrow G \twoheadrightarrow \mathsf{C}$, and a natural transformation fitting into the square these functors form

$$\begin{tikzcd} F \downarrow G \arrow[r, two heads] \arrow[d, two heads] \arrow[dr, phantom, "\Rightarrow"] & \mathsf{C} \arrow[d, "G"] \\ \mathsf{B} \arrow[r, "F"'] & \mathsf{A} \end{tikzcd}$$

(ii) Describe the universal property of $F \downarrow G$. In particular, explain what data determines a functor $\mathsf{E} \to F \downarrow G$ and explain what data determines a natural transformation $\begin{tikzcd} \mathsf{E} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & F \downarrow G \end{tikzcd}$.

Exercise 7. Suppose that $F\colon \mathsf{C} \to \mathsf{D}$ and $G\colon \mathsf{D} \to \mathsf{C}$ define an equivalence of categories. Prove that F is left adjoint to G and also that G is left adjoint to F by constructing the units and counits of these adjunctions from the invertible 2-cells that comprise the equivalence of categories.[4]

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[4] A proof written in this manner will apply more generally in any 2-category, not only in the 2-category of categories, functors, and natural transformations.