Math 616: Algebraic Topology
Problem Set 5
due: May 3, 2016

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Exercise 1. Let $F \colon \mathsf{A} \to \mathsf{B}$ be an additive functor between abelian categories. Its **prolongation** is the functor $F \colon \mathsf{ChA} \to \mathsf{ChB}$ that you get by just applying F degreewise.

(i) Prove that the prolongation preserves chain homotopy equivalences.
(ii) Suppose further that F is exact. Prove that its prolongation preserves quasi-isomorphisms.

Exercise 2. Recall:

- $S^n \in \mathsf{Ch}_R$ is the chain complex with the R-module R in degree n and zeros elsewhere.
- $D^n \in \mathsf{Ch}_R$ is the chain complex with the R-module R in degrees n and $n-1$, with identity differential, and zeros elsewhere.

Let $A \in \mathsf{Ch}_R$ be a chain complex of R-modules.

(i) Define an injective chain map $S^{n-1} \to D^n$.
(ii) Explain precisely what data is needed to define a chain map $D^n \to A$.[1]
(iii) Explain precisely what data is needed to define a chain map $S^n \to A$.

Exercise 3. Prove that a chain map $f \colon A \to B$ has the right lifting property against the map $0 \to D^n$

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & A \arrow[d, "f"] \\ D^n \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

if and only if $f_n \colon A_n \to B_n$ is a surjection at the level of underlying sets (every element of B_n is in the image of f_n).

Exercise 4. Prove that if a chain map $f \colon A \to B$ has the right lifting property against the map $S^n \to D^{n+1}$

$$\begin{tikzcd} S^n \arrow[r] \arrow[d] & A \arrow[d, "f"] \\ D^{n+1} \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

then

(i) $H_n f \colon H_n A \to H_n B$ is a monomorphism and
(ii) $H_{n+1} f \colon H_{n+1} A \to H_{n+1} B$ is an epimorphism.

Exercise 5. Consider a functor $F \colon \mathsf{M} \to \mathsf{N}$ whose domain is a model category and whose codomain is a category with a specified class of weak equivalences. Prove that if F carries trivial fibrations between fibrant objects to weak equivalences then F preserves all weak equivalences between fibrant objects.

[1] That is, tell me that chain maps $D^n \to A$ are in (natural) bijection with elements of some set; your task is to identify this set.

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