Math 616: Algebraic Topology
Problem Set 4
due: April 19, 2016

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Exercise 1. For any topological space X, there is a space X^I whose points are continuous functions $I \to X$ (i.e., **paths in** X) topologized using the compact open topology.[1]

(i) Show that the "constant path function" $X \to X^I$ is a homotopy equivalence.[2]
(ii) Show that the "endpoint evaluation function" $X^I \to X \times X$ is a fibration.[3]
(iii) Show that each "single endpoint evaluation function" $X^I \to X$ is both a fibration and a homotopy equivalence.[4]

Exercise 2. For any continuous function $f\colon X \to Y$ the **mapping path space** Nf is defined by the following pullback:

$$\begin{tikzcd} Nf \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y^I \arrow[d, two heads] \\ X \times Y \arrow[r, "f \times 1"] & Y \times Y \end{tikzcd}$$

Use Nf to factor $f\colon X \to Y$ as a homotopy equivalence followed by a fibration.

Exercise 3. Let $\mathcal{L} = {}^{\boxslash}\mathcal{R}$ be the class of maps that have the left lifting property against some class of maps $\mathcal{R}$.

(i) Prove that every isomorphism is in $\mathcal{L}$.
(ii) If $f\colon X \to Y$ and $g\colon Z \to W$ are in $\mathcal{L}$, show that the coproduct (disjoint union) $f \sqcup g\colon X \sqcup Y \to Z \sqcup W$ is in $\mathcal{L}$.

Exercise 4. Prove that there are weak factorization systems on the category of sets:

(i) whose left class is the monomorphisms and whose right class is the epimorphisms.
(ii) whose left class is the epimorphisms and whose right class is the monomorphisms.

Exercise 5. In fact, the category of sets admits exactly six weak factorization systems:

(i) (monomorphisms, epimorphisms)
(ii) (epimorphisms, monomorphisms)
(iii) (isomorphisms, all maps)
(iv) (all maps, isomorphisms)
(v) (all maps with non-empty domain, isos plus maps with empty domain)

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[1] Feel to assume that you are working with a cartesian closed category of spaces, if you know what this means.

[2] It's okay to define the homotopy inverse and witnessing homotopies set-theoretically, without proving these functions are continuous.

[3] Similarly, it's okay to define lifts set-theoretically, without verifying that these functions are continuous.

[4] Hint: the homotopy equivalences satisfy the 2-of-3 property.

(vi) (all monos with non-empty domain, epis plus maps with empty domain)

Which pairs of these define model category structures on the category of sets?

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