Math 616: Algebraic Topology

Problem Set 1[1]

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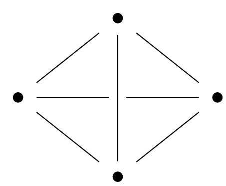
Exercise 1. A *tetrahedron* is a geometric simplicial complex of dimension 2 with four vertices, six edges, and four faces:



Define its associated chain complex of abelian groups and compute its homology in all degrees.

Exercise 2. Recall a morphism $f\colon A \to B$ in an additive category is a *monomorphism* if and only if $fa = 0$ implies $a = 0$ for any morphism $a\colon X \to A$. Dually, f is an *epimorphism* if and only if $bf = 0$ implies $b = 0$ for all $b\colon B \to X$. Use universal properties to prove:

(i) The kernel of a morphism is always a monomorphism.

$$\ker f \rightarrowtail A \xrightarrow{f} B$$

(ii) The cokernel of a morphism is always an epimorphism.[2]

$$A \xrightarrow{f} B \twoheadrightarrow \operatorname{coker} f$$

Exercise 3. Suppose A is a category with *finite direct sums*. This means that A has a zero object, has binary coproducts, has binary products, and for any pair of objects A and B the canonical morphism

$$A \sqcup B \xrightarrow{\begin{pmatrix} 1_A & 0 \\ 0 & 1_B \end{pmatrix}} A \times B$$

is an isomorphism. Prove that each hom-set $\hom(A, B)$ canonically inherits the structure of a commutative monoid $(\hom(A, B), +, 0)$ in such a way that composition is bilinear: i.e., so that given

$$A \xrightarrow{f} B \underset{g'}{\overset{g}{\rightrightarrows}} C \xrightarrow{h} D$$

then $h(g + g')f = hgf + hg'f$.

Exercise 4. Suppose A is a category that is enriched over abelian groups. Prove that if A has binary products then A has binary direct sums: i.e., if there exist maps

$$A \xleftarrow{\pi_A} A \times B \xrightarrow{\pi_B} B$$

[1]Problems labelled n^* are optional (fun!) challenge exercises that will not be graded.

[2]If you want to argue by duality that's fine, but explain what an argument by duality is.

satisfying the universal property of the product, then there also exist maps

$$A \xrightarrow{\iota_A} A \times B \xleftarrow{\iota_B} B$$

satisfying the universal property of the coproduct and so that

$$\begin{pmatrix} \pi_A \\ \pi_B \end{pmatrix} \cdot \begin{pmatrix} \iota_A & \iota_B \end{pmatrix} = \begin{pmatrix} 1_A & 0 \\ 0 & 1_B \end{pmatrix}$$

Exercise 5. Show that for a functor $F\colon \mathsf{A} \to \mathsf{B}$ between abelian categories the following are equivalent:

(i) For each pair of objects $A, A' \in \mathsf{A}$, the function $\hom(A, A') \to \hom(FA, FA')$ is a group homomorphism.
(ii) F preserves direct sums.

Exercise 6. Given a short exact sequence of chain complexes

$$0 \longrightarrow A_\bullet \rightarrowtail B_\bullet \twoheadrightarrow C_\bullet \longrightarrow 0$$

prove that if any two of these chain complexes is exact so is the third.

Exercise 7*. Extend the definition of (i) a chain complex and (ii) its homology from Mod_R to any abelian category.[3] Argue that if A is abelian, then the category $\mathsf{Ch}(\mathsf{A})$ is again abelian.

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[3] That is, recall the definition of a chain complex, recall the definition of homology, and observe that all of the pieces can be interpreted in any abelian category.