Math 601: Algebra
Problem Set 8
due: November 9, 2017

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Exercise 1.

(i) Consider a short exact sequence of R-modules

$$0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0$$

Prove that M is Noetherian if and only if N and M/N are Noetherian.

(ii) Let R be a Noetherian ring and let $I \subset R$ be an ideal. Prove that R/I is a Noetherian ring and explain the connection, if any, between this question and (i).

Exercise 2. Prove that the ring of continuous functions $[0,1] \to \mathbb{R}$ is not Noetherian.

Exercise 3. Consider the subring

$$\mathbb{Z}[\sqrt{-5}] := \{a + bi\sqrt{5} \mid a, b \in \mathbb{Z}\} \subset \mathbb{C}.$$

(i) Express $\mathbb{Z}[\sqrt{-5}]$ as a finite type $\mathbb{Z}$-algebra.

(ii) Prove that $\mathbb{Z}[\sqrt{-5}]$ is a Noetherian integral domain.

(iii) Consider the norm function $N\colon \mathbb{Z}[\sqrt{-5}] \to \mathbb{N}$ defined by $N(a + bi\sqrt{5}) = a^2 + 5b^2$ and describe its properties — i.e., what sort of homomorphism is it?

(iv) Find all units in $\mathbb{Z}[\sqrt{-5}]$.

(v) Prove that $2, 3, 1 + i\sqrt{5}, 1 - i\sqrt{5}$ are all irreducible elements but none of these are prime.

(vi) Conclude that $\mathbb{Z}[\sqrt{-5}]$ has factorizations into irreducibles but is not a UFD.

Exercise 4. Let R be a UFD and let $I \subset R$ be a non-zero ideal. Prove that every descending chain of principal ideals containing I must stabilize.

Exercise 5. The height h of a prime ideal $\mathfrak{p} \subset R$ is the length of the maximal chain of prime ideals

$$0 = \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_h = \mathfrak{p}.$$

Prove that if R is a UFD then every prime ideal of height 1 is principal.

Exercise 6.

(i) Prove that any Euclidean domain R admits a Euclidean valuation $v\colon R\backslash\{0\} \to \mathbb{N}$ with the property that $v(ab) \geq v(b)$ for all non-zero $a, b \in R$.

(ii) With respect to the valuation defined in (i) prove that associate elements have the same valuation and that units have minimum valuation. Are the converses true?

Exercise 7. A **discrete valuation** on a field $\Bbbk$ is a surjective homomorphism of abelian groups $v\colon (\Bbbk^\times, \times) \to (\mathbb{Z}, +)$ so that $v(a+b) \geq \min\{v(a), v(b)\}$ for all $a, b \in \Bbbk^\times$ so that $a + b \in \Bbbk^\times$.

(i) Prove that $R = \{a \in \Bbbk^\times \mid v(a) \geq 0\} \cup \{0\}$ is a subring of $\Bbbk$.

(ii) Prove that R is a Euclidean domain.

(iii) Rings of the form of (i) are called **discrete valuation rings**. Find an example of a discrete valuation ring.

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