Math 601: Algebra
Problem Set 7[1]
due: November 1, 2017

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Exercise 1*. Let G be a group with $|G| = pq$, where $p < q$ are primes.

(i) If $q \not\equiv 1 \bmod p$ prove that G is cyclic.
(ii) If G is non-abelian prove that $q \equiv 1 \bmod p$.
(iii) In both cases (i) and (ii) determine the lattice of subgroups of G.

Exercise 2*. Assume G is a non-abelian group with $|G| = pq$, where $p < q$ are primes.

(i) Prove that G is not a simple group.
(ii) Find the number of elements of G that have each possible order.
(iii) Find the number and sizes of the conjugacy classes of elements in G.

Exercise 3 (clarification about direct sums)**.**

(i) Let R and S be rings. Recall that $R \oplus S \cong R \times S$ as underlying abelian groups. Conclude that $R \oplus S$ *is* a ring with componentwise multiplication.
(ii) Verify that $R \oplus S$ fails to satisfy the universal property of the *coproduct* in the category of rings.
(iii) Let $\{R_i\}_{i \in I}$ be a family of rings indexed by an infinite set i. Explain why the abelian group $\oplus_{i \in I} R_i$ is *not* a ring with componentwise multiplication.

Exercise 4 (Aurel's exercise)**.** Let R be a commutative ring. Prove that $\oplus_n R$ admits ring structures that do not arise from the additive group isomorphism $\oplus_n R \cong R[x]/(f(x))$ for some monic polynomial of degree n.

Exercise 5. A ring R is **Boolean** if $a^2 = a$ for every $a \in R$.

(i) Endow the set of subsets of a set S with a Boolean ring structure.
(ii) Prove that every non-zero Boolean ring is commutative and has characteristic 2.

Exercise 6. Let $f(x) \in R[x]$ be a monic polynomial. Prove that for every $g(x) \in R[x]$ there exist unique polynomials $q(x)$ and $r(x)$ so that

$$g(x) = f(x) \cdot q(x) + r(x)$$

and $\deg r(x) < \deg q(x)$.[2]

Exercise 7. Prove that if $\Bbbk$ is a field then $\Bbbk[x]$ is a PID.

Exercise 8. Let R be a commutative ring.

(i) Let $a_1, \ldots, a_n \in R$. Express the quotient ring

$$R[x_1, \ldots, x_n]/(x_1 - a_1, \ldots, x_n - a_n)$$

as a quotient of the ring R.

[1] If you aren't 100% confident that you know how to solve the problems n^* you should really write up the solutions, but we aren't going to grade them.

[2] Adopt the convention that the degree of the zero polynomial is $-\infty$.

(ii) Let $f_1(x), \ldots, f_n(x) \in R[x]$ and $a \in R$. Express the quotient ring

$$R[x]/(f_1(x), \ldots, f_n(x), x - a)$$

as a quotient of the ring R.

Exercise 9. Let K be a compact topological space and let R denote the ring of continuous functions $K \to \mathbb{R}$, with addition and multiplication defined pointwise in $\mathbb{R}$.

(i) For $p \in K$ define $I_p = \{f \in R \mid f(p) = 0\}$. Prove that $I_p \subset R$ is a maximal ideal.

(ii) Prove that if $f_1, \ldots, f_n \in R$ have no common zeros then $(f_1, \ldots, f_n) = R$. [Hint: consider $f_1^2 + \cdots f_n^2$.]

(iii) Prove that every maximal ideal of R is of the form I_p for some $p \in K$. [Hint: this is where you use compactness.]

In summary, $p \mapsto I_p$ defines a bijection between K and the set of maximal ideals of R.

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