Math 601: Algebra
Problem Set 6
due: October 18, 2017

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Exercise 1. Let

$$G = G_0 \supsetneq G_1 \supsetneq G_2 \supsetneq \cdots \supsetneq G_n = \{e\}$$

be a normal series. Define a sequence of group extensions involving the G_i and the quotients $H_i = G_i/G_{i+1}$ that realize G as an extension of groups that are extensions of extensions of extensions of (etc) the groups H_i and the trivial group $\{e\}$.

Exercise 2. Let N and H be any groups equipped with a homomorphism $\phi \colon H \to \mathrm{Aut}(N)$. Their **semidirect product** is the group $N \rtimes_\phi H$ whose underlying set is $N \times H$ and in which multiplication is defined by the formula:

$$(n_1, h_1) \cdot (n_2, h_2) := (n_1 \phi_{h_1}(n_2), h_1 h_2).$$

(i) Define an injective homomorphism $N \rightarrowtail N \rtimes_\phi H$ whose image is a normal subgroup.
(ii) Define a surjective homomorphism $N \rtimes_\phi H \twoheadrightarrow H$ and a splitting homomorphism $H \hookrightarrow N \rtimes_\phi H$.
(iii) Prove that the sequence

$$1 \longrightarrow N \longrightarrow N \rtimes_\phi H \longrightarrow H \longrightarrow 1$$

defined by (i) and (ii) is split exact.
(iv) Identifying N and H with their images in $N \rtimes_\phi H$ prove that $N \cap H = \{e\}$.
(v) Identifying N and H with their images in $N \rtimes_\phi H$, prove that $\phi \colon H \to \mathrm{Aut}(N)$ agrees with the homomorphism defined by conjugating N by H in $N \rtimes_\phi H$.
(vi) Identifying N and H with their images in $N \rtimes_\phi H$, prove that $NH \cong N \rtimes_\phi H$.

Exercise 3. Express D_{2n} as a semi-direct product of cyclic groups.

Exercise 4. Let N and H be any groups equipped with a homomorphism $\phi \colon H \to \mathrm{Aut}(N)$. Prove that if $N \rtimes_\phi H$ is commutative then $N \rtimes_\phi H \cong N \times H$.

Exercise 5. Let A and B be two abelian groups and consider only extensions of A by B that are again abelian groups. Let $\mathrm{Ext}(A, B)$ be the set of isomorphism classes of extensions of A by B, where two extensions are isomorphic if there exists an isomorphism

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d, "\cong"] & A \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & B \arrow[r] & E' \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

commuting with the inclusions of B and projections to A. Prove that $\mathrm{Ext}(A, B)$ is an abelian group with the sum of two extensions

$$0 \longrightarrow B \xrightarrow{i} E \xrightarrow{p} A \longrightarrow 0$$

$$0 \longrightarrow B \xrightarrow{j} F \xrightarrow{q} A \longrightarrow 0$$

defined to be the quotient group $\ker(p-q)/\operatorname{im}(i-j)$, where $(p-q)\colon E\oplus F \to A$ is defined by $(p,q)(e,f) = p(e) - q(f)$ and $(i-j)\colon B \to E \oplus F$ is defined by $(i-j)(b) = (i(e), -j(e))$.

Exercise 6. Recall that for any abelian groups A and B the set $\operatorname{Hom}(A,B)$ of homomorphisms from A to B is again an abelian group. Moreover, for any group homomorphisms $\phi\colon B \to B'$ and $\psi\colon A' \to A$ there is an induced group homomorphism $\operatorname{Hom}(A,B) \to \operatorname{Hom}(A',B')$ defined by pre- and post-composition.[1] In categorical language, $\operatorname{Hom}(-,-)\colon \mathsf{Ab}^{\mathrm{op}} \times \mathsf{Ab} \to \mathsf{Ab}$ defines a functor that is *contravariant* in its first variable and *covariant* in its second variable.

Show similarly that $\operatorname{Ext}(-,-)\colon \mathsf{Ab}^{\mathrm{op}} \times \mathsf{Ab} \to \mathsf{Ab}$ is contravariant in its first variable and covariant in its second variable. That is, given homomorphisms $\phi\colon B \to B'$ and $\psi\colon A' \to A$ define functions

$$\operatorname{Ext}(A,B) \to \operatorname{Ext}(A',B) \qquad \text{and} \qquad \operatorname{Ext}(A,B) \to \operatorname{Ext}(A,B')$$

that compose to define a single function $\operatorname{Ext}(A,B) \to \operatorname{Ext}(A',B')$.[2]

Exercise 7.

(i) Let R and S be any rings. Define a ring structure on their cartesian product $R \times S$ and prove that this defines a product in the category of rings.

(ii) It is not a coincidence that the product of rings and the product of groups may be defined by defining an appropriate ring or group structure on the product of their underlying sets.[3] Speculate why this construction defines the product for any class of "algebraic" objects and speculate why it is *not* possible to define the coproduct in these categories by defining appropriate structures on the disjoint union of the underlying sets.

Exercise 8. A **Laurent polynomial** with coefficients in a ring R is a finite formal sum $\sum_{k\in\mathbb{Z}} r_k x^k$ where $r_k \in R$ and only finitely many of the coefficients are non-zero. The ring of Laurent polynomials is denoted by $R[x,x^{-1}]$. Define an isomorphism of rings $R[\mathbb{Z}] \cong R[x,x^{-1}]$.[4]

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[1] It suffices to consider the pre- and post-compositions separately, as these commute.

[2] Importantly, these functions are group homomorphisms, preserving the extension sum defined in Exercise 5. But this might be a pain to check, so you don't have to write up a proof of the homomorphism property if you don't want to.

[3] In both cases, these may be characterized as the unique ring/group structures making the projection functions into homomorphisms.

[4] It's okay to use this isomorphism to define multiplication and addition in either of the two rings under consideration, but spell out the definition of these operations in notation for polynomials and in notation for group rings.