Math 601: Algebra

Problem Set 5
due: October 11, 2017

Emily Riehl

Exercise 1.

(i) Let $H \subset G$ be a subgroup of index 2. Prove that H is normal in G.
(ii) Argue that if G has a simple subgroup of index 2, then it has a composition series of length 2.

Exercise 2. Prove that any non-zero group homomorphism whose domain is simple is injective.

Exercise 3. Classify all finite simple *abelian* groups. (Hint: start by thinking about the order of the group.)

Exercise 4. Prove that no p-group of order at least p^2 is simple.

Exercise 5. Show that any group of order mp with p prime and $1 < m < p$ is not simple.

Exercise 6. Find an example of non-isomorphic groups of order greater than 4 that have the same composition factors.

Exercise 7. Give a constructive proof that $G_1 \times G_2$ has a composition series if and only if G_1 and G_2 do.[1]

Exercise 8. Prove that A_n has a trivial center for all n.[2]

Exercise 9. Prove that the sign of a permutation $\sigma \in S_n$ is equal to the determinant of the $n \times n$-matrix $A = (a_{i,j})_{1 \leq i,j \leq n}$ with $a_{i,j} = 1$ iff $i \cdot \sigma = j$ and 0 otherwise. You may use any theorems from linear algebra that you like, provided you state them clearly.

Dept. of Mathematics, Johns Hopkins Univ., 3400 N Charles St, Baltimore, MD 21218
E-mail address: `eriehl@math.jhu.edu`

[1] A non-constructive proof of this is immediate if G_1 and G_2 are finite, so please do *not* assume finiteness.

[2] The meaning of "trivial" goes through a phase change as n increases: A_1, A_2, and A_3 are abelian, so the center is the entire group. For $n > 3$, A_n is non-abelian and its center is the trivial group.