Math 601: Algebra
Problem Set 4[1]
due: October 4, 2017

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Exercise 1. Define a presentation for the dihedral group D_{2n} with two generators r and s and justify the relations you enumerate by arguing that every element of the dihedral group has a unique representation as $r^m s^n$ where $m, n \geq 0$ and are each less than the orders of r and s respectively.

Exercise 2. Let $\phi\colon K \to G$ be a group homomorphism and let N be the smallest normal subgroup containing the image of ϕ. Prove that $\pi\colon G \twoheadrightarrow G/N$ defines the **cokernel** of ϕ by showing that it satisfies the universal property speciifed in §8.6 (or in class).

Exercise 3. Consider a surjective homomorphism[2] $G \twoheadrightarrow K$ in the category of groups. Either prove that every such homomorphism admits a right inverse homomorphism or produce an example that proves that this need not be the case.

Exercise 4. Let G be a group and let A be a set.

(i) Given a group homomorphism $\rho\colon G \to \mathrm{Aut}(A)$, define a function of two variables $\alpha\colon G \times A \to A$, the "action of G on A," so that the diagrams

$$\begin{tikzcd} G \times G \times A \arrow[r, "\cdot \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times \alpha"'] & G \times A \arrow[d, "\alpha"] \\ G \times A \arrow[r, "\alpha"'] & A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "e \times \mathrm{id}"] \arrow[dr, "\mathrm{id}"'] & G \times A \arrow[d, "\alpha"] \\ & A \end{tikzcd}$$

commute in Set.

(ii) Given a function $\alpha\colon G \times A \to A$ so that the diagrams displayed above commute, define a function $\rho\colon G \to \mathrm{End}(A)$ and prove that it (a) lands in the subset $\mathrm{Aut}(A) \subset \mathrm{End}(A)$ and (b) defines a group homomorphism.

Exercise 5. Use the universal property of $\mathbb{Z}/n$ to argue that to define the action of $\mathbb{Z}/n$ on $A \in \mathsf{C}$ it is necessary and sufficient to define an automorphism $f\colon A \to A$ of order n, i.e., so that $f^{\circ n} = \mathrm{id}_A$. If G is presented by a set of generators S modulo relations R, what data is needed to describe a G-action?

Exercise 6. Explore the connection between *right* actions of G and *left* actions of G^{op} by solving exercise 9.3 from your book on a piece of scratch paper or on the chalkboard. (You do not need to write anything up.)

Exercise 7. Define an explicit bijection (i.e., don't simply argue that these sets have the same cardinality) between the set G/H of left cosets and the set $H\backslash G$ of right cosets of any subgroup $H \subset G$. Hint: you might use the isomorphism $G \cong G^{\mathrm{op}}$ established in Exercise 6 and the canonical transitive actions on these sets.

Exercise 8.

[1]Problems labelled n^* are optional (fun!) challenge exercises that will not be graded.

[2]FYI the epimorphisms in the category of groups are exactly the surjective homomorphisms; you don't have to prove this.

(i) What is the center of D_{2n}?
(ii) What is the center of S_n?

Exercise 9. A Rubik's cube is built from 26 little cubes called *cubies*; the expected 27th cubie at the very center of the cube is missing.[3] The *Rubik's cube group* is generated by six elements of order four R, L, F, B, U, D which act on the Rubik's cube by performing one counterclockwise rotation of the right, left, front, bottom, upwards, and downards faces, respectively.

(i) What are the fixed points of the Rubik's cube action?
(ii) What are the orbits?
(iii) What is the stabilizer of the upper, right, front corner cubie?
(iv) What is the stabilizer of the upper, front edge cubie?
(v) Verify the orbit-stabilizer theorem and the class formula ($26 = \sum ??$) for this action.[4]

Exercise 9*. Prove that the free group on 26 generators $a, b, c, \ldots, z$ modulo pronunciation in English is trivial.[5]

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[3] For the purposes of this problem we will consider the cubies to be unoriented.

[4] It's possible that these lemmas are much harder to precisely prove than I realize, in which case feel free to only sketch the answers to each question.

[5] Alternatively, google "homophonic quotients of free groups."