Math 601: Algebra
Problem Set 3
due: September 27, 2017

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Exercise 1. Let A be a set and let G be a group. Prove that any function $f\colon A \to G$ extends uniquely to a group homomorphism $\phi_f\colon FA \to G$ whose domain is the free group generated by the elements of A along the specified inclusion of the generating elements $\eta\colon A \to FA$.[1]

Exercise 2. Use the universal property of the free group to prove that the free group construction defines a functor $F\colon \mathsf{Set} \to \mathsf{Grp}$ that is a **left adjoint**[2] to the forgetful functor $U\colon \mathsf{Grp} \to \mathsf{Set}$:

(i) For any function $f\colon A \to B$ define a homomorphism $Ff\colon FA \to FB$ so that the diagram of underlying functions commutes:

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f"'] & UFA \arrow[d, dashed, "UFf"] \\ B \arrow[r, "\eta_B"] & UFB \end{tikzcd}$$

(ii) Argue that the homomorphism Ff defined in (i) is the *unique* homomorphism making the square commute.

(iii) Use (ii) to prove that $F1_A = 1_{FA}$, that is F carries the identity function to the identity homomorphism.

(iv) Use (ii) to prove that for any composable pair of functions $f\colon A \to B$ and $g\colon B \to C$, $F(gf) = Fg \circ Ff$.

(v) Conclude that F defines a functor $F\colon \mathsf{Set} \to \mathsf{Grp}$.

(vi) For any set A and group G define a (natural[3]) bijection between functions $A \to UG$ and homomorphisms $FA \to G$.

$$\mathsf{Set}(A, UG) \cong \mathsf{Group}(FA, G).$$

This is what it means for the functor F to be **left adjoint** to the functor U.[4]

Exercise 3. Define the **free product** $G * H$ of groups G and H to be the quotient of $F(UG \coprod UH)$ — the free group on the underlying sets of G and H — modulo the smallest normal subgroup N containing the elements $e_G e_H^{-1}$ and $k_1 k_2 k_3^{-1}$ for

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[1] This version of the universal property of the free group can be stated more precisely using the forgetful functor $U\colon \mathsf{Grp} \to \mathsf{Set}$: for any set A and group G prove that any function $f\colon A \to UG$ induces a unique homomorphism $\phi_f\colon FA \to G$ so that the diagram $\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "f"'] & UFA \arrow[d, "U\phi_f"] \\ & UG \end{tikzcd}$ commutes in Set; here I'm adopting the standard categorical convention of eliding parentheses when applying a functor to an object, writing simply "UG" for the underlying set of G.

[2] You don't need to know what this means.

[3] This is a technical term which you can look up if you are curious, but you are not responsible for justifying the "naturality" of your construction. If you wish to do so, please do this on a separate sheet of paper with a note to Sarah saying "you don't have to read this."

[4] The same categorical argument implies that the free abelian group construction defines a functor $F\colon \mathsf{Set} \to \mathsf{Ab}$ that is left adjoint to $U\colon \mathsf{Ab} \to \mathsf{Set}$.

any triple of elements all in G or all in H so that $k_1k_2 = k_3$.[5] Prove that $G * H := F(UG \coprod UH)/N$ is the **coproduct** of G and H in the category of groups by direct verification of this universal property using this explicit construction of $G * H$.

Exercise 4. Let A and B be *abelian* groups. In this context, the product $A \times B$ is usually denoted by $A \oplus B$ and referred to as the **direct sum**. Recall from Exercise 9 on Problem Set 2 that there exist canonical homomorphisms $\iota_A \colon A \to A \oplus B$ and $\iota_B \colon B \to A \oplus B$ so that this data define a **coproduct** in the category Ab of abelian groups. Note that the homomorphisms exist whether or not A and B are abelian, but the universal property only holds in the case where you are mapping into an abelian group. (You don't have to write anything.)

Exercise 5. Prove that the **free product** $\mathbb{Z} * \mathbb{Z}$ is the **free group** on two generators and prove that the **direct sum** $\mathbb{Z} \oplus \mathbb{Z}$ is the **free abelian group** on two generators. Or, if you would rather solve both of these problems at once, prove that for any category C with an "underlying set" functor $U \colon \mathsf{C} \to \mathsf{Set}$ the coproduct of two copies of the free object of C on one generator defines the free object of C on two generators (with a universal properties defined as in Exercise 1), whenever such coproducts exist. Note that $\mathbb{Z}$ is both the free group and the free abelian group on a single generator.

Exercise 6. Read §II.5.4 if you haven't already and think about how to generalize Exercise 5 to the following statements: the free group on A is the A-indexed free product $*_A \mathbb{Z} = \mathbb{Z}^{*A}$ while the free abelian group on A is the A-indexed direct sum $\oplus_A \mathbb{Z} = \mathbb{Z}^{\oplus A}$. (You don't have to write anything.)

Exercise 7. The **commutator subgroup** $[G, G] \subset G$ of a group G is the subgroup generated by elements of the form $aba^{-1}b^{-1}$ for $a, b \in G$.

(i) Prove that $[G, G]$ is a normal subgroup of G.
(ii) Prove that the quotient group $G/[G, G]$ is abelian. This quotient is called the **abelianization** of the group G.
(iii) Use the universal property of the quotient group to show that $G \mapsto G/[G, G]$ defines a functor $(-)^{\mathrm{ab}} \colon \mathsf{Group} \to \mathsf{Ab}$.
(iv) Use the universal property of the quotient group to show that for any abelian group A and any group G there is a (natural) bijection between group homomorphisms $G \to A$ and $G^{\mathrm{ab}} \to A$.[6]
(v) What is another name for the abelianization of the free group FS on a set S? Answer this question by saying that a certain triangle of functors commutes.

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[5] Defining N to be the normal subgroup generated by these elements is equivalent to defining an equivalence relation $\sim$ on the elements of $F(UG \coprod UH)$ generated by the relations $e_G \sim e_H$ and $k_1k_2 \sim k_3$. Elements of $G * H$ are then words in the elements of G and H modulo relations that say that $e_G = e_H$ and any products $k_1k_2 = k_3$ that hold in G or in H also hold in $G * H$.

[6] This says that the functor $(-)^{\mathrm{ab}} \colon \mathsf{Group} \to \mathsf{Ab}$ is left adjoint to the inclusion $\mathsf{Ab} \hookrightarrow \mathsf{Group}$.