Math 601: Algebra
Problem Set 2
due: September 20, 2017

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Exercise 1. The group $\mathbb{Z}$ is the *free group on one generator*. What is its universal property?[1]

Exercise 2. Prove that there are no non-zero group homomorphisms between the Klein four group and $\mathbb{Z}/7$.

Exercise 3.

- Define an element of order d in S_n for any $d < n$.
- For which n is S_n abelian? Give a proof or supply a counterexample for each $n \geq 1$.

Exercise 4. Is the product of cyclic groups cyclic? If so, give a proof. If not, find a counterexample.

Exercise 5. Prove that if A and B are abelian groups, then $A \times B$ satisfies the universal property of the *coproduct* in the category Ab of abelian groups. Explain why the commutativity hypothesis is necessary.

Exercise 6. Prove that the function $g \mapsto g^{-1}$ defines a homomorphism $G \to G$ if and only if G is abelian.[2]

Exercise 7.

(i) Fix an element g in a group G. Prove that the conjugation function $x \mapsto gxg^{-1}$ defines a homomorphism $\gamma_g \colon G \to G$.
(ii) Prove that the function $g \mapsto \gamma_g$ defines a homomorphism $\gamma \colon G \to \text{Aut}(G)$. The image of this function is the subgroup of *inner automorphisms* of G.
(iii) Prove that γ is the zero homomorphism if and only if G is abelian.

Exercise 8. We have seen that the set of homomorphisms $\hom(B, A)$ between two abelian groups is an abelian group with addition defined pointwise in A. In particular $\text{End}(A) := \hom(A, A)$ is an abelian group under pointwise addition.

(i) Is the set $\text{Aut}(A)$ of automorphisms of A a subgroup of $\text{End}(A)$ under pointwise addition?
(ii) We have seen that the set $\text{Aut}(A)$ is a group under composition of homomorphisms. More generally, $\text{End}(A)$ is a monoid under composition of homomorphisms.[3] How does the monoid structure $(\text{End}(A), \circ)$ interact with the abelian group structure $(\text{End}(A), +)$?[4]

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[1] I recommend solving this problem before it is spoiled by Monday's class.

[2] For a generic group, the function $g \mapsto g^{-1}$ does define a homomorphism, and in fact an isomorphism, $G^{\text{op}} \to G$ from the opposite group of G to G. In particular, any group is isomorphic to its opposite group.

[3] $\text{End}(A)$ is a monoid, rather than a group, because homomorphisms need not have composition inverses.

[4] It is possible to give a complete description of this interaction, but it makes use of terminology that we have not encountered yet. You are welcome to do so if you like, but I'll be satisfied if you instead write down a few explicit properties connecting the operations $+$ and $\circ$ and their respective identity elements.

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