Math 601: Algebra
Problem Set 10[1]
due: November 29, 2017

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Exercise 1. Let R be any commutative ring.[2]

(i) Describe an explicit bijection $\mathrm{Hom}_R(R^{\oplus m}, R^{\oplus n}) \cong \mathsf{Mat}_R(m,n)$ between homomorphisms and $n \times m$ matrices with coefficients in R by explaining how to define a homomorphism $R^{\oplus m} \to R^{\oplus n}$ given a matrix A.

(ii) Let A be an $n \times m$ matrix. For any $1 \leq i \leq m$ and $1 \leq j \leq n$ the composite

$$R \xrightarrow{\iota_i} R^{\oplus m} \xrightarrow{A} R^{\oplus n} \xrightarrow{\pi_j} R$$

of the inclusion of the i-th component, the homomorphism corresponding to A, and the projection onto the j-th component defines an element of $\mathrm{Hom}_R(R,R) \cong R$. What is it?

(iii) For any d-dimensional column vector $\vec{a}$ and d-dimensional row vector $\vec{b}$ the composite

$$R \xrightarrow{\vec{a}} R^{\oplus d} \xrightarrow{\vec{b}} R$$

defines an element of $\mathrm{Hom}_R(R,R) \cong R$. What is it?

(iv) Combining facts (i), (ii), and (iii) explain why the composite homomorphism

$$R^{\oplus m} \xrightarrow{A} R^{\oplus d} \xrightarrow{B} R^{\oplus n}$$

is given by the product $B \cdot A$ of the $n \times d$ matrix B with the $d \times m$ matrix A.

Exercise 2.

(i) Let R be a commutative ring and let $\mathfrak{m}$ be any maximal ideal with quotient field $\Bbbk = R/\mathfrak{m}$. Prove that for any free module $F \cong R^{\oplus B}$ over R, $F/\mathfrak{m}F \cong \Bbbk^B$ as $\Bbbk$-vector spaces.

(ii) Conclude that for any commutative ring if $R^{\oplus A} \cong R^{\oplus B}$ then $A \cong B$, so that free modules over any commutative ring have a well-defined rank.

Exercise 3. Let R be the ring of linear endomorphisms of a $\Bbbk$-vector space V, that is $R := \mathrm{End}_{\mathsf{Vect}_\Bbbk}(V)$.

(i) Is R a commutative ring?

(ii) Prove that $\mathrm{End}_{\mathsf{Vect}_\Bbbk}(V \oplus V) \cong R^{\oplus 4}$ as an R-module.

(iii) Observe that in the case $V = \Bbbk^{\oplus \mathbb{N}}$, $V \cong V \oplus V$. Conclude that free R-modules do *not* have a well-defined rank, unlike the case of Exercise 2.

Exercise 4. Classify homomorphism between free modules that do not necessarily have finite rank. What is $\mathrm{Hom}_R(R^{\oplus A}, R^{\oplus B})$ for any sets A and B?[3]

Exercise 5. If M is a finitely generated free R-module a choice of basis for M is given by an isomorphism $M \cong R^{\oplus S}$.

[1]Problems labelled $n*$ are optional (fun!) challenge exercises that will not be graded.

[2]Commutativity is not in fact necessary.

[3]Just write a few sentences unpacking the explicit definition. The description isn't meant to be especially nice.

(i) Explain how a homomorphism $\phi\colon M \to N$ of finitely generated free R-modules may be encoded by a matrix by choosing bases for M and for N.
(ii) Let S and S' be two bases for a finitely generated free module M of cardinality n. Define the corresponding change of basis matrix as an element of $GL_n(R)$.
(iii) Say two matrices $P, Q \in \mathsf{Mat}_R(m,n)$ are **equivalent** if and only if they represent the same homomorphism $R^{\oplus m} \to R^{\oplus n}$. Redefine this equivalence relation in terms of matrices.
(iv) Prove that P is equivalent to Q if P can be obtained from Q by a sequence of elementary row operations.

Exercise 6*. Let R be a Euclidean domain.

(i) Prove that two $n \times m$ matrices over R are equivalent in the sense of Exercise 5 if and only if they can be linked by a sequence of elementary row operations,
(ii) Prove that every $n \times m$ matrix is equivalent to a matrix $A = (a_{jk})_{1\leq j\leq n, 1\leq k\leq m}$ in **Smith normal form**, with the only non-zero entries including on the diagonal a_{ii} with $i \leq m, n$ and $a_{ii} \mid a_{i+1,i+1}$ for all i.

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