Math 601: Algebra

Problem Set 1[1]

due: September 13, 2017

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Exercise 1. A morphism $i\colon A \to B$ admits a *left inverse* or a *retraction* if there exists a morphism $r\colon B \to A$ so that $r \circ i = 1_A$. In this case, i is called a *split monomorphism*.

(i) Prove that split monomorphisms are in fact monomorphisms.
(ii) State the dual definition of a *split epimorphism* in any category.
(iii) Prove that any morphism that is both a split monomorphism and an epimorphism is an isomorphism. Conclude by duality that any morphisms that is both a split epimorphism and a monomorphism is an isomorphism.

Exercise 2. Consider a commutative triangle of morphisms

$$\begin{tikzcd} A \arrow[rr, "h"] \arrow[dr, "f"'] & & C \\ & B \arrow[ur, "g"'] & \end{tikzcd}$$

(i) Prove that if f and g are monomorphisms so is their composite h.
(ii) Prove that if h is a monomorphism then so is f.
(iii) Find an example to show that it is possible for h to be a monomorphism while g is not.

Exercise 3. Prove that the collection of isomorphisms in any category C define a *subcategory* of C, with the same objects and with composition and identities defined by restricting these operations from C.[2] This category is called the *maximal subgroupoid* or sometimes the *groupoid core* of C.

Exercise 4. Let $\sim$ be an equivalence relation on a set A. Prove that there exists a bijection between the set of functions from A to B that respect the equivalence relation and the set of functions from $A_{/\sim}$ to B. Explain the connection between this bijection and the universal property of the canonical projection map $\pi\colon A \twoheadrightarrow A_{/\sim}$.[3]

Exercise 5. Suppose that P and Q are two objects of C that satisfy the universal property of being the coproduct of $A, B \in \mathsf{C}$. Prove that $P \cong Q$.

Exercise 6.

(i) Define the *product* and the *coproduct* of a collection $(A_i)_{i \in I}$ of objects $A_i \in \mathsf{C}$ indexed by a set I so that when I has cardinality 2, this generalizes the binary products and coproducts defined in class.

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[1] Problems labelled n^* are optional (fun!) challenge exercises that will not be graded.

[2] If you think this exercise is boring here's a challenge: give a clear and complete answer in as few sentences as possible.

[3] This universal property holds for any relation $\sim$ on A, regardless of whether $\sim$ is an equivalence relation. In this more general context the fibers of the quotient map $\pi\colon A \twoheadrightarrow A_{/\sim}$ partition A into the set of equivalence classes for the equivalence relation *generated by* $\sim$, which is the smallest equivalence relation containing $\sim$.

(ii) What do these definitions specialize to in the case where $I = \emptyset$?

Exercise 7. Sketch a proof that any category with binary products and a terminal object has all *finite products*, i.e., has all products indexed by a finite set. What is necessary for a category to have all finite coproducts?

Exercise 8. Recall that a category with at most one morphism in each hom-set is a *preorder*, a set P with a reflexive transitive relation $\leq$. In the corresponding category the objects are the elements of P and there is a morphism $x \to y$ iff $x \leq y$. What must be true of $(P, \leq)$ for this category to have binary products?[4] What must be true of $(P, \leq)$ for this category to have binary coproducts? A poset $(P, \leq)$ with finite products and coproducts is called a *lattice*.

Exercise 9*. There are countably-infinitely many prisoners, each clearly labelled by a natural number, held in an execution chamber. Momentarily each will be given either a red or blue hat, after which point each prisoner will be able to see everyone else's hat but not their own. After the hats are distributed the prisoners are not permitted to communicate in any way. Then, on the count of three, each must simultaneously guess the color of their own hat and every prisoner who guesses incorrectly will be killed. Before the hats are distributed, the prisoners may collaborate on a strategy for guessing hat colors. Prove that it is possible for them to guarantee that no matter the distribution of red and blue hats, only finitely many prisoners will be killed.

What does this have to do with Exercise 4?

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[4] That is, how would a set theorist express the axiom that $(P, \leq)$ has binary products?