Math 411: Honors Algebra I
Final
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Guidelines

• I will ask you to answer at least three of the questions on the following list (and I get to pick). You'll then answer orally or using the chalkboard. I expect each exam to take 20 minutes or less.

• You are welcome (indeed encouraged) to quote any theorems from class that you like, but I may then ask you to state them (these statements can be approximate — "the canonical decomposition theorem says that any function can be factored as a quotient followed by an inclusion" — they don't have to be perfect). I may also ask you to define the terms that appear in the questions below (eg "what is a subgroup?").

• You are welcome to consult handwritten (but not typed or photocopied) notes to refresh your memory before explaining the solution, but I will deduct points if you read your solution from your notes. The point is I want the answers to be *internalized* (i.e., understood), not memorized or read.

• You are welcome to work with others in the class in preparing for the exam, but please write up any notes you wish to bring on your own. You'll ultimately be examined on how well *you* understand the solutions.

• I will do my best to ask everyone a follow-up question that you do not know the answer to. This is just for fun, to push you to think about something new. DO NOT PANIC when this happens.

• You're welcome to ask me questions too, if you like, or not. (This is not a part of the exam.) The point of an oral final is to give everyone a chance to have a mathematical conversation with me one-on-one.

Questions

1. By Cayley's theorem, every subgroup is isomorphic to a subgroup of some symmetric group. Is $\mathbb{Z}/3 \times \mathbb{Z}/4$ isomorphic to a subgroup of S_4?

2. Describe the 3-sylow subgroups of S_6. What are they isomorphic to? How many are there?

3. Let G act on a set S and let $x \in S$. Show that the stabilizer G_x defines a subgroup of G.

4. Describe the conjugacy classes of elements of S_4 and explain how this calculation tells you that S_3 is not a normal subgroup of S_4.

5. What is the universal property of $\mathbb{Z}$ as a group? What is the universal property of $\mathbb{Z}$ as a ring? Explain why the answers to each of these questions are different.

6. Show, for any element a in any ring, that the additive inverse of a is equal to the product of a with the additive inverse of the multiplicative identity.

7. Let $\phi\colon R \to S$ be a ring homomorphism. Does $\ker\phi$ define a normal subgroup of the underlying group $(R, +, 0)$ of the ring? Why or why not?

8. Any ring homomorphism $\phi\colon R \to S$ must *preserve* units. That is, if u is a unit in R then $\phi(u)$ is a unit in S. Do ring homomorphisms also *reflect* units? That is, if $\phi(v)$ is a unit in S, is v a unit in R? Prove this or find a counterexample.

9. Any integer has a unique factorization into powers of distinct prime integers. State conditions in terms of the prime factorizations

$$a = \pm p_1^{r_1} \cdots p_n^{r_n} \qquad \text{and} \qquad b = \pm q_1^{s_1} \cdots q_m^{s_m}$$

of $a, b \in \mathbb{Z}$ that correspond to the conditions

(i) $(a) \subset (b)$
(ii) $(b) \subset (a)$
(iii) $(a) = (b)$

on ideals of $\mathbb{Z}$.

10. The ring of Gaussian integers $\mathbb{Z}[i]$ is defined to be the quotient ring $\mathbb{Z}[x]/(x^2+1)$. Apply the Canonical Decomposition Theorem to identify $\mathbb{Z}[i]$ with a subring of $\mathbb{C}$.