Math 411: Honors Algebra I
Problem Set 9
due: November 13, 2019

Emily Riehl

Exercise 1. Prove that a group homomorphism $\phi\colon G \to H$ is injective if and only if $\ker\phi = \{e\}$.

Proof. Clearly $\phi(e) = e$ so if ϕ is injective then $\ker\phi = \{e\}$. Conversely if $\ker\phi = \{e\}$ and $\phi(g) = \phi(g')$ then $\phi(g^{-1}g') = e$ so $g^{-1}g' = e$ so $g = g'$. ☐

Exercise 2. How many elements of S_8 are conjugate to $(12)(345)(678)$?

Proof. We've proved that elements of the symmetric group are conjugate iff they have the same cycle shape. So the question asks how many permutations in S_8 have the cycle shape of $(12)(345)(678)$. There are 28 ways to pick a 2-cycle and then 10 ways to partition the remaining six elements into two sets of 3, plus two ways to pick each 3 cycle for

$$28 \times 10 \times 2 \times 2 = 1120$$

total permutations (out of $8! = 40320$). ☐

Exercise 3. The **center** of a group G is the subgroup

$$Z(G) = \{g \in G \mid \forall x \in G, gx = xg\}$$

made up of all elements that commute with all other elements of g.

(i) Prove that $Z(G)$ is a subgroup.
(ii) Prove that $g \in Z(G)$ if and only if the conjugacy class of g contains a single element.
(iii) Prove that $Z(G)$ is normal in G.
(iv) Prove that any subgroup $H \subset Z(G)$ is a normal subgroup of G.

Proof. (i) If $g \in Z(G)$ then for all $h \in G$, $(g^{-1}h)^{-1} = h^{-1}g = gh^{-1} = (hg^{-1})^{-1}$. Thus $g^{-1}h = hg^{-1}$ so $g^{-1} \in Z(G)$. It's easy to verify that products of elements in the center are in the center, so $Z(G)$ is a subgroup.

(ii) Since $gh = hg$ iff $hgh^{-1} = g$, we see that g commutes with all elements of h iff the conjugacy class of g is a singleton.

(iii) By (i) and (ii) $Z(G)$ is a union of conjugacy classes of its elements and as such is a normal subgroup.

(iv) By (ii) any subgroup of $Z(G)$ is a union of conjugacy classes of its elements and as such is normal. ☐

Exercise 4. Recall a group is *simple* if it has no non-trivial normal subgroups. Prove that any non-zero group homomorphism whose domain is a simple group is injective.

Proof. By Exercises 1, if a homomorphism is not injective it has a non-zero kernel. But this kernel must be a normal subgroup, so if the group is simple the kernel must be the entire group, and the homomorphism is then the zero homomorphism. ☐

Exercise 5. Show that any group of order mp with p prime and $1 < m < p$ is not simple.

Proof. By the first Sylow theorem, any group of order mp has a subgroup of order p. By the second Sylow theorem, all such subgroups are conjugate, but by the third Sylow theorem the number of such subgroups divides m and is 1 mod p. Since $m < p$ this tells us there is a single subgroup of order p, which we conclude is a normal subgroup. ☐

Exercise 6. Classify all finite simple *abelian* groups. (Hint: start by thinking about the order of the group.)

Proof. Any subgroup of an abelian group is normal, so an abelian group is simple if and only if it has no non-trivial subgroups. By Cayley's theorem if a prime p divides the order of a group then there is a subgroup of order p. So an abelian group can be simple if and only if it has prime order. Thus the finite simple abelian groups are the groups $\mathbb{Z}/p$ for p a prime. ☐

Dept. of Mathematics, Johns Hopkins Univ., 3400 N Charles St, Baltimore, MD 21218
E-mail address: `eriehl@math.jhu.edu`