Math 411: Honors Algebra I
Problem Set 8
due: November 6, 2019

Emily Riehl

Exercise 1. Let G be a group.

(i) Prove that if $x, y \in G$ are conjugate, then x and y have the same order.
(ii) Prove that the order of any conjugacy class of elements in G divides the order of the group G.
(iii) Let $N \lhd G$ be a normal subgroup. Prove that N is the union of the conjugacy classes of its elements.

Exercise 2. Let I denote the **icosahedral group**, the group of symmetries of the icosahedron (or equally, by duality of the platonic solids, of the dodecahedron). In problem set 5, you discovered that $|I| = 60$.

(i) Calculate the orders of the elements of I. In particular, determine how many elements have each order and describe the resulting partition:
$$60 \text{ elements } = 1 \text{ element of order one } + \ldots ?$$
(ii) Calculate the conjugacy classes of elements of I and describe the resulting partition of $60 = |I|$ (the class equation). Explain why this partition refines the partition you found in (i).[1]
(iii) Prove that I is a **simple group**: that is, show that I has no non-trivial normal subgroups.

The result in (iii) is useful for identifying I. The group I acts on the set of cubes inscribed inside the dodecahedron. Since there are five such cubes, this action defines a homomorphism $I \to S_5$. Since I is a simple group, the kernel of this homomorphism must either be I or $\{e\}$. Since the action is non-trivial it's the latter, and consequently I may be identified with a subgroup of S_5 of order 60. To find this subgroup, we consider the composite homomorphism $I \to S_5 \to \mathbb{Z}/2$ where the second map is the **sign homomorphism**, sending even cycles to $[0]$ and odd cycles to $[1]$. If this homomorphism were surjective, the first isomorphism theorem would tell us that $\mathbb{Z}/2$ is isomorphic to a quotient group I/N, where $N \lhd I$ is a normal subgroup of order 30. But such a subgroup doesn't exist, so $I \to S_5 \to \mathbb{Z}/2$ must be the zero homomorphism. Thus I is contained in the kernel of the sign homomorphism $S_5 \to \mathbb{Z}/2$, which is the alternating group A_5. Since $I \subset A_5$ and both groups have the same order, we conclude that $I \cong A_5$.

Exercise 3. Find the center of D_{2n}. [Hint: the answer depends on whether n is even or odd.]

Exercise 4. Prove that the center of S_n is trivial for $n \geq 3$.

Exercise 5. If $H \subset G$ is a subgroup its conjugate subgroups are the subgroups of the form
$$gHg^{-1} = \{ghg^{-1} \mid h \in H\}$$
for some $g \in G$.

(i) Prove that gHg^{-1} is a subgroup of G.
(ii) Define a bijective group homomorphism $H \to gHg^{-1}$.
(iii) The group G acts on the set of subgroups of G by conjugation: the action of a group element $g \in G$ on a subgroup $H \subset G$ is defined by $H \mapsto gHg^{-1} \subset G$. Rephrase the condition of H being a normal subgroup in terms of the orbits of this action.

Exercise 6. Prove that S_p, where p is prime, is generated by just two permutations: the transposition (12) and $(12 \cdots p)$.

Exercise 7. Find the formula for the size of the conjugacy class of a permutation of any given cycle shape in S_n.

Exercise 8. Prove that any normal subgroup of S_4 must have order $1, 4, 12$, or 24.

Dept. of Mathematics, Johns Hopkins Univ., 3400 N Charles St, Baltimore, MD 21218
E-mail address: `eriehl@math.jhu.edu`

[1]Hint: Exercise 1 will help you identify which elements are likely to be conjugate and which are likely not to be conjugate. It's okay to wave your hands a bit in the proofs as long as you state clearly what you are guessing is true.