Math 411: Honors Algebra I
Problem Set 8
due: November 6, 2019

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Exercise 1. Let G be a group.

(i) Prove that if $x, y \in G$ are conjugate, then x and y have the same order.
(ii) Prove that the order of any conjugacy class of elements in G divides the order of the group G.
(iii) Let $N \triangleleft G$ be a normal subgroup. Prove that N is the union of the conjugacy classes of its elements.

Proof. (i) Suppose $x = gyg^{-1}$, which is to say that x is the image of y under the homomorphism $g(-)g^{-1} \colon G \to G$. Since this homomorphism is a group isomorphism, it preserves the order of elements and the result follows.

(ii) A conjugacy class of elements is an orbit under the conjugacy action of G on itself. Thus, by the orbit stabilizer theorem, the order of the conjugacy class divides the order of the group.

(iii) Recall that $N \subset G$ is normal if for all $g \in G$,

$$gNg^{-1} = \{gng^{-1} \mid n \in N\} \subset N.$$

This says directly that for each $n \in N$, the conjugacy class of n is a subset of N. So N must be the union of the conjugacy classes of its elements. □

Exercise 2. Let I denote the **icosahedral group**, the group of symmetries of the icosahedron (or equally, by duality of the platonic solids, of the dodecahedron). In problem set 5, you discovered that $|I| = 60$.

(i) Calculate the orders of the elements of I. In particular, determine how many elements have each order and describe the resulting partition:

$$60 \text{ elements } = 1 \text{ element of order one } + \ldots ?$$

(ii) Calculate the conjugacy classes of elements of I and describe the resulting partition of $60 = |I|$ (the class equation). Explain why this partition refines the partition you found in (i).[1]
(iii) Prove that I is a **simple group**: that is, show that I has no non-trivial normal subgroups.

The result in (iii) is useful for identifying I. The group I acts on the set of cubes inscribed inside the dodecahedron. Since there are five such cubes, this action defines a homomorphism $I \to S_5$. Since I is a simple group, the kernel of this homomorphism must either be I or $\{e\}$. Since the action is non-trivial it's the latter, and consequently I may be identified with a subgroup of S_5 of order 60. To find this subgroup, we consider the composite homomorphism $I \to S_5 \to \mathbb{Z}/2$ where the second map is the **sign homomorphism**, sending even cycles to $[0]$ and odd cycles to $[1]$. If this homomorphism were surjective, the first isomorphism theorem would tell us that $\mathbb{Z}/2$ is isomorphic to a quotient group I/N, where $N \triangleleft I$ is a normal subgroup of order 30. But such a subgroup doesn't exist, so $I \to S_5 \to \mathbb{Z}/2$ must be the zero homomorphism. Thus I is contained in the kernel of the sign homomorphism $S_5 \to \mathbb{Z}/2$, which is the alternating group A_5. Since $I \subset A_5$ and both groups have the same order, we conclude that $I \cong A_5$.

Proof. (i) The icosahedral group contains 60 elements, each of which can be understood as some "rotation" of the icosahedron. One of these is the identity element, of order 1. For each of the 6 opposing pairs of vertices, there are 4 rotations about the axis between them of order 5, resulting in 24 elements of order 5. Similarly, for each of the 15=30/2 opposing pairs of edges, there is one rotation of order 2 about the axis between their midpoints, for 15 elements of order 2. Finally, for each of the 20/2 = 10 opposing pairs of faces, there are 2 rotations of order 3 about the axis between the centers of these faces, for 20 elements of order 3. Since $1 + 24 + 15 + 20 = 60$, we've accounted for all of the elements in the icosahedral group.

(ii) By Exercise 1 conjugate elements must have the same order, so the partition into conjugacy classes must refine the partition $60 = 1 + 15 + 20 + 24$ found above. Moreover, since the orders of conjugacy classes divide the order of the group, we see immediately that the rotations of order 5 cannot all be conjugate. We first argue that the 20 elements of order 3, which we call "face rotations," are all conjugate. Note that a "face rotation," viewed from outside the icosahedron, always

[1] Hint: Exercise 1 will help you identify which elements are likely to be conjugate and which are likely not to be conjugate. It's okay to wave your hands a bit in the proofs as long as you state clearly what you are guessing is true.

entails a counter-clockwise rotation of $2\pi/3$ about one of the faces and a clockwise rotation of $2\pi/3$ about the other face. The element $g \in I$ that conjugates one face rotation to any other face rotation is any symmetry that moves the clockwise rotating face for one rotation to the clockwise rotating face for the other rotation: apply g, then apply the second rotation, then apply g^{-1}, and this will recover the first face rotation. A similar (but easier) argument shows that any of the edge rotations are conjugate, where the conjugating element is any one that sends the fixed edge to the fixed edge.

We call the remaining 24 rotations of order 5 "vertex rotations." These split up into two classes. One class entails a counter-clockwise rotation of $2\pi/5$ around one of the vertices together with a clockwise rotation of $2\pi/5$ about its antipode. The other class entails a counter-clockwise rotation of $4\pi/5$ about one of the vertices paired with a clockwise rotation of $4\pi/5$ about its antipode. By the same argument, all members of the former class are conjugate, as are all members of the latter class. Thus the class equation has the form

$$60 = 1 + 15 + 20 + 12 + 12.$$

(iii) By Exercise 1, normal subgroups are unions of conjugacy classes. Subgroups of course also contain the identity. But there is no way to add 1 to some non-empty subset of the numbers 15, 20, 12, and 12 to get a divisor of 60 unless you include them all. This proves that I has no non-trivial normal subgroups. □

Exercise 3. Find the center of D_{2n}. [Hint: the answer depends on whether n is even or odd.]

Proof. Recall

$$D_{2n} = \langle x, y \mid x^2 = e = y^n, xyx = y^{n-1}.$$

In particular x and y do not commute and when n is odd any of the other rotations y^j do not commute with x since $xy^jx = y^{n-j}$. Since x can be any reflection this proves that D_{2n} has a trivial center when n is odd.

When $n = 2k$ is even almost the same argument works with one exception: the 180 degree rotation y^k has $xy^kx = y^k$ so y is in the center. In this case the center is $\mathbb{Z}/2$ generated by the 180 degree rotation. □

Exercise 4. Prove that the center of S_n is trivial for $n \geq 3$.

Proof. If $\sigma \in S_n$ is a non-identity permutation there must be some i so that $\sigma(i) = j$ where $i \neq j$. Choose some $k \neq i, j$. Then $(jk)\sigma(jk)$ sends i to k while σ sends i to j. Thus σ does not commute with (jk) which means that σ is not in the center. □

Exercise 5. If $H \subset G$ is a subgroup its conjugate subgroups are the subgroups of the form

$$gHg^{-1} = \{ghg^{-1} \mid h \in H\}$$

for some $g \in G$.

(i) Prove that gHg^{-1} is a subgroup of G.
(ii) Define a bijective group homomorphism $H \to gHg^{-1}$.
(iii) The group G acts on the set of subgroups of G by conjugation: the action of a group element $g \in G$ on a subgroup $H \subset G$ is defined by $H \mapsto gHg^{-1} \subset H$. Rephrase the condition of H being a normal subgroup in terms of the orbits of this action.

Proof. For (i) it suffices to show that for ghg^{-1} and gkg^{-1} in gHg^{-1} (i.e., for $h, k \in H$) we have $ghg^{-1}(gkg^{-1})^{-1} \in gHg^{-1}$. This works out to $ghg^{-1}gk^{-1}g = ghk^{-1}g^{-1}$ which is in gHg^{-1} since $hk^{-1} \in H$.

For (ii) the bijection is given by the map $h \mapsto ghg^{-1}$. Its inverse is given by $(ghg^{-1}) \mapsto g^{-1}(ghg^{-1})g = h$.

For (iii), H is normal iff it is a fixed point for the conjugation action. □

Exercise 6. Prove that S_n is generated by just two permutations: the transposition (12) and $(12 \cdots n)$.

Proof. Since S_n is generated by the adjacent transpositions, it suffices to show that the cycles $(ii+1)$ can be obtained as products of $(12 \cdots n)$ and (12). But you can check that $(12 \cdots n)^{n-i+1}(12)(12 \cdots p)^{i-1} = (ii+1)$. □

Exercise 7. Find the formula for the size of the conjugacy class of a permutation of any given cycle shape in S_n.

Proof. We proved in class that conjugacy classes correspond to cycle shapes. So suppose there are k_1 1-cycles, k_2 2-cycles, ...k_n n-cycles. (Note most of these numbers are probably zero.)

There are n total slots in this set of cycles that can be filled in $n!$ possible ways. But several different ways of filling the blanks correspond to the same element of S_n. For instance each of the k_j j-cycles can be written in j different ways (cyclically permuting the entries) and each of the k_j j-cycles can be written in $k_j!$ different orders. So the correct size of the conjugacy class is

$$\frac{n!}{\prod_{i=1}^{n} i^{k_i} \cdot k_i!}.$$

□

Exercise 8. Prove that any normal subgroup of S_4 must have order 1, 4, 12, or 24.

Proof. A normal subgroup must be a union of conjugacy classes, including the identity conjugacy class. The conjugacy classes have the following sizes:

- the class of e has 1 element
- the class of (12) has 6 elements
- the class of $(12)(34)$ has 3 elements
- the class of (123) has 8 elements
- the class of (1234) has 6 elements

Normal subgroups must also be subgroups so their order in particular must divide the order of S_4, which is 24. The possible sums that divide 24 and include 1 are

$$1, \quad 1+3, \quad 1+3+8, \quad 1+3+6+6+8,$$

and that's it. You can check that these do each correspond to subgroups. □

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