Math 411: Honors Algebra I
Problem Set 7[1]
due: October 30, 2019

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Exercise 1. Define a presentation for the dihedral group D_{2n} with two generators r and s. Then justify the relations you enumerate by arguing that every element of the dihedral group has a unique representation as $r^m s^n$ where $m, n \geq 0$ and are each less than the orders of r and s respectively and then sketch a proof that you can reduce any word in the elements r and s can be reduced to a word of the form $r^m s^n$ by iteratively applying the relations you enumerate.[2]

Exercise 2. Let G be a group and let A be a set.

(i) Given a group homomorphism $\rho\colon G \to \mathrm{Aut}(A)$, define a function of two variables $\alpha\colon G \times A \to A$, the "action of G on A," so that the diagrams

$$\begin{array}{ccc} G \times G \times A & \xrightarrow{\cdot \times \mathrm{id}} & G \times A \\ {\scriptstyle \mathrm{id}\times\alpha}\downarrow & & \downarrow{\scriptstyle \alpha} \\ G \times A & \xrightarrow[\alpha]{} & A \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{e \times \mathrm{id}} & G \times A \\ & {\scriptstyle \mathrm{id}}\searrow & \downarrow{\scriptstyle \alpha} \\ & & A \end{array}$$

commute in Set.

(ii) Given a function $\alpha\colon G \times A \to A$ so that the diagrams displayed above commute, define a function $\rho\colon G \to \mathrm{End}(A)$ and prove that it (a) lands in the subset $\mathrm{Aut}(A) \subset \mathrm{End}(A)$ and (b) defines a group homomorphism.

Exercise 3.

(i) Use the universal property of $\mathbb{Z}/n$ to argue that to define the action of $\mathbb{Z}/n$ on a set A it is necessary and sufficient to define an automorphism $f\colon A \to A$ of order n, i.e., so that $f^{\circ n} = \mathrm{id}_A$.

(ii) If G is presented by a set of generators S modulo relations R, what data is needed to describe a G-action?

Exercise 4. The group $\mathbb{Z}/2$ acts on $\mathbb{C}$ by complex conjugation.

(i) Use Exercise 3 to explain what is meant by the previous sentence.

(ii) Any group action on a set defines a partition of that set into orbits. Describe the resulting partition of the complex plane into orbits.

(iii) An element $z \in \mathbb{C}$ is **fixed** by the complex conjugation action if its orbit is a singleton. What are the fixed points of this action?

Exercise 5. A Rubik's cube is built from 26 little cubes called *cubies*; the expected 27th cubie at the very center of the cube is missing.[3] The *Rubik's cube group* is generated by six elements of order four R, L, F, B, U, D which act on the Rubik's cube by performing one counterclockwise rotation of the right, left, front, bottom, upwards, and downards faces, respectively. The Rubik's cube action identifies the Rubik's cube group with a subgroup of S_{26}.

(i) Any group action on a set defines a partition of that set into orbits. Describe the resulting partition of the set of 26 cubies into orbits.

(ii) A cubie is **fixed** by the Rubik's cube action if its orbit is a singleton. What are the fixed points of the Rubik's cube action?

Exercise 6. Let $H \subset G$ be a subgroup. Then G acts on the set of left cosets G/H by left multiplication as discussed in class.

(i) What is the orbit of the left coset H?

(ii) What is the stabilizer of the left coset H?

(iii) What is the orbit of a generic left coset gH?

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[1] Problems labelled n^* are optional (fun!) challenge exercises that will not be graded.

[2] In class we defined the relations to be elements that generate the kernel of the canonical homomorphism $\phi\colon F\{r,s\} \to D_{2n}$ in the sense that the smallest normal subgroup that contains these elements is $\ker\phi$. But for the purposes of applying relations to reduce words, it might be easiest to present your relations as equations between elements of $F\{r,s\}$. For instance, if r and s commuted (which in this case they do not), this would be expressed by the relation $rs = sr$, which would say that $rsr^{-1}s^{-1}$ is in the kernel of ϕ (which, again, is not the case here).

[3] For the purposes of this problem we will consider the cubies to be unoriented. However, a more precise description of the Rubik's cube group would take orientations of the cubies into account.

(iv) What is the stabilizer of a generic left coset gH?

Exercise 7*. Prove that the free group on 26 generators $a, b, c, \ldots, z$ modulo pronunciation in English is trivial.[4]

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[4]Alternatively, google "homophonic quotients of free groups."