Math 411: Honors Algebra I
Problem Set 7[1]
due: October 30, 2019

Emily Riehl

Exercise 1. Define a presentation for the dihedral group D_{2n} with two generators r and s. Then justify the relations you enumerate by arguing that every element of the dihedral group has a unique representation as $r^m s^n$ where $m, n \geq 0$ and are each less than the orders of r and s respectively and then sketch a proof that you can reduce any word in the elements r and s can be reduced to a word of the form $r^m s^n$ by iteratively applying the relations you enumerate.[2]

Proof. Let r be a basic rotation and let s be one of the reflections. Then $r^n = e$ and $s^2 = e$. We can check that rs is again a reflection so $(rs)^2 = e$, so $rs = (rs)^{-1} = sr^{n-1}$. Thus D_{2n} has a presentation

$$D_{2n} = \langle r, s \mid r^n = e, s^2 = e, rs = sr^{n-1} \rangle.$$

Now given any element σ of D_{2n} either it defines an orientation-preserving action on the n-gon (keeping the vertices ordered clockwise, say) or is an orientation-reversing one. In the first case the element is r^k for some $0 \leq k < n$. In the second case $\sigma \cdot s$ is orientation-preserving so $\sigma \cdot s = r^j$ for some $0 \leq j < n$ and thus $\sigma = r^j s$. There are $2n$ elements of D_{2n} and $2n$ elements in the set $\{r^m s^k \mid m < n, k < 2\}$ so we have found the unique representation for each one. □

Exercise 2. Let G be a group and let A be a set.

(i) Given a group homomorphism $\rho\colon G \to \mathrm{Aut}(A)$, define a function of two variables $\alpha\colon G \times A \to A$, the "action of G on A," so that the diagrams

$$\begin{array}{ccc} G \times G \times A & \xrightarrow{\cdot \times \mathrm{id}} & G \times A \\ {\scriptstyle \mathrm{id}\times\alpha}\downarrow & & \downarrow{\scriptstyle \alpha} \\ G \times A & \xrightarrow[\alpha]{} & A \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{e \times \mathrm{id}} & G \times A \\ & {\scriptstyle \mathrm{id}}\searrow & \downarrow{\scriptstyle \alpha} \\ & & A \end{array}$$

commute in Set.

(ii) Given a function $\alpha\colon G \times A \to A$ so that the diagrams displayed above commute, define a function $\rho\colon G \to \mathrm{End}(A)$ and prove that it (a) lands in the subset $\mathrm{Aut}(A) \subset \mathrm{End}(A)$ and (b) defines a group homomorphism.

Proof. (i) Define $\alpha(g, a) = \rho(g)(a)$, the application of the function $\rho(g)$ to the element a. We have

$$\alpha(h, \alpha(g, a)) = \alpha(h, \rho(g, a)) = \rho(h)(\rho(g)(a)) = \rho(hg)(a) = \alpha(hg, a) \tag{1}$$

since ρ is a homomorphism and similarly $\alpha(e, a) = \rho(e)(a) = a$ since $\rho(e)$ is the identity function.

(ii) Now given α define $\rho(g)\colon A \to A$ to be the function $a \mapsto \alpha(g, a)$, that is $\rho(g)(-) = \alpha(g, -)$. We can see that $\rho(g)$ is an automorphism since it has an inverse function: namely $\rho(g^{-1})$. We can see that it defines a group homomorphism because $\rho(h)\rho(g) = \rho(hg)$ by undoing the calculation (1) above.

□

Exercise 3.

(i) Use the universal property of $\mathbb{Z}/n$ to argue that to define the action of $\mathbb{Z}/n$ on a set A it is necessary and sufficient to define an automorphism $f\colon A \to A$ of order n, i.e., so that $f^{\circ n} = \mathrm{id}_A$.

(ii) If G is presented by a set of generators S modulo relations R, what data is needed to describe a G-action?

Proof. By the universal property of $\mathbb{Z}/n$ to define a homomorphism $\mathbb{Z}/n \to G$ it is necessary and sufficient to find an element $g \in G$ so that $g^n = e$. So to define $\mathbb{Z}/n \to \mathrm{Aut}(A)$ we need find $f \in \mathrm{Aut}(A)$ with order n, i.e., so that $f^{\circ n} = \mathrm{id}_A$.

More generally if G is presented by generators S modulo relations R, to define $G \to \mathrm{Aut}(A)$ we must define an automorphism $f_s\colon A \to A$ for every $s \in S$ so that for every relation the corresponding composites of these automorphisms are equal: eg the relation $ts = st$ would correspond to the composition condition $f_t \circ f_s = f_s \circ f_t$. More formally, to say

[1]Problems labelled n^* are optional (fun!) challenge exercises that will not be graded.

[2]In class we defined the relations to be elements that generate the kernel of the canonical homomorphism $\phi\colon F\{r, s\} \to D_{2n}$ in the sense that the smallest normal subgroup that contains these elements is $\ker\phi$. But for the purposes of applying relations to reduce words, it might be easiest to present your relations as equations between elements of $F\{r, s\}$. For instance, if r and s commuted (which in this case they do not), this would be expressed by the relation $rs = sr$, which would say that $rsr^{-1}s^{-1}$ is in the kernel of ϕ (which, again, is not the case here).

that G is generated by $s_1, \dots, s_k \in G$ modulo relations $r_1, \dots, r_n \in F\{s_1, \dots, s_k\}$ means that $G \cong F\{s_1, \dots, s_k\}/\langle r_1, \dots, r_n\rangle$, where $\langle r_1, \dots, r_n\rangle \subset F\{s_1, \dots, s_k\}$ is the smallest normal subgroup containing these elements. By the universal property of quotient groups to define a homomorphism $F\{s_1, \dots, s_k\}/\langle r_1, \dots, r_n\rangle \to \mathrm{Aut}(A)$ is to define a homomorphism $\phi \colon F\{s_1, \dots, s_k\} \to \mathrm{Aut}(A)$ containing $r_1, \dots, r_n$ in its kernel. By the universal property of free groups, this amounts to specifying automorphism $f_{s_1}, \dots, f_{s_k}$ of A so that for each word r_i, the corresponding composite of the automorphisms is the identity function. □

Exercise 4. The group $\mathbb{Z}/2$ acts on $\mathbb{C}$ by complex conjugation.

(i) Use Exercise 3 to explain what is meant by the previous sentence.
(ii) Any group action on a set defines a partition of that set into orbits. Describe the resulting partition of the complex plane into orbits.
(iii) An element $z \in \mathbb{C}$ is **fixed** by the complex conjugation action if its orbit is a singleton. What are the fixed points of this action?

Proof. To define an action of $\mathbb{Z}/2$ on $\mathbb{C}$ we need to define $\mathbb{Z}/2 \to \mathrm{Aut}(\mathbb{C})$. By the previous problem it is enough to define an automorphism $f \colon \mathbb{C} \to \mathbb{C}$ that squares to the identity, i.e., so that $f \circ f = \mathrm{id}$. Complex conjugation $f(x+iy) = x - iy$ has this property.

Each real number $x + i0$ is fixed by the conjugation action, so $\{x\}$ is its own orbit. The other orbits have order 2 and contain complex conjugate pairs $\{x + iy, x - iy\}$. This also answer question (iii). □

Exercise 5. A Rubik's cube is built from 26 little cubes called *cubies*; the expected 27th cubie at the very center of the cube is missing.[3] The *Rubik's cube group* is generated by six elements of order four R, L, F, B, U, D which act on the Rubik's cube by performing one counterclockwise rotation of the right, left, front, bottom, upwards, and downards faces, respectively. The Rubik's cube action identifies the Rubik's cube group with a subgroup of S_{26}.

(i) Any group action on a set defines a partition of that set into orbits. Describe the resulting partition of the set of 26 cubies into orbits.
(ii) A cubie is **fixed** by the Rubik's cube action if its orbit is a singleton. What are the fixed points of the Rubik's cube action?

Proof. The 12 edges are in one obit, the eight edges are in another orbit, and each of the center pieces is fixed. □

Exercise 6. Let $H \subset G$ be a subgroup. Then G acts on the set of left cosets G/H by left multiplication as discussed in class.

(i) What is the orbit of the left coset H?
(ii) What is the stabilizer of the left coset H?
(iii) What is the orbit of a generic left coset gH?
(iv) What is the stabilizer of a generic left coset gH?

Proof. (i) The action of G on G/H is transitive so the orbit of H is the entire set G/H.

(ii) The stabilizer of H under this action is $H \subset G$ because $gH = H$ iff $g \in H$.

(iii) The transitivity of the action also says the orbit of gH is the entire set G/H.

(iv) The stabilizer of gH is the subgroup $gHg^{-1} \subset H$ (see problem 3 on problem set 8 for more about this subgroup). To see this note $kgH = gH$ if and only if $g^{-1}kg \in H$ so the stabilizer is given by those k so that $k \in gHg^{-1}$. All elements of the form ghg^{-1} stabilize gH.

□

Exercise 7*. Prove that the free group on 26 generators $a, b, c, \dots, z$ modulo pronunciation in English is trivial.[4]

DEPT. OF MATHEMATICS, JOHNS HOPKINS UNIV., 3400 N CHARLES ST, BALTIMORE, MD 21218
E-mail address: `eriehl@math.jhu.edu`

[3] For the purposes of this problem we will consider the cubies to be unoriented.

[4] Alternatively, google "homophonic quotients of free groups."